A new approach to Gaussian beams on a sphere: theory and application to long-period surface wave propagation

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Accepted 1989 February 3. Received 1989 February 1; in original form 1988 October 12.

SUMMARY
The subject of this paper is the application of the Gaussian beam method to the propagation of surface waves on a laterally heterogeneous, spherical earth taking fully into account the curvature of the Earth's surface. This is achieved by constructing a ray centred coordinate system on general curved surfaces. The application of this method to a spherical shell leads to a ray centred coordinate system adjusted to the spherical geometry. Solving the elastodynamic equation in this coordinate system by a perturbation ansatz yields two interesting results. (1) The well-known parabolic differential equation for the zeroth order displacement follows directly from a solvability condition imposed on the right-hand side of an inhomogeneous differential equation arising from the perturbation ansatz. (2) The dynamic ray-tracing system which is derived from the parabolic equation is a special case of a general dynamic ray-tracing system valid for arbitrary curved surfaces which can be derived by perturbing the ray equations. It is shown that the geometry of the surface enters only by its intrinsic curvature into the dynamic ray-tracing system. Numerical computations of rays and synthetic seismograms based on this new approach for the model M84C of Woodhouse & Dziewonski show that there can occur considerable deviations of the ray-paths from great circles leading to visible amplitude anomalies in the synthetic seismograms. Comparison of synthetic seismograms with selected data recorded at the Black Forest Observatory (BFO) indicate that the model M84C is still too smooth to yield satisfactory agreement of the synthetics with the data.

Key words: defocusing, focusing, Gaussian beam method, lateral heterogeneities, ray-centred coordinates

1 INTRODUCTION
The study of long-period surface waves ($T > 100$ s) has shown that in many cases the propagation behaviour of surface waves differs considerably from what is to be expected on a laterally homogeneous Earth. The observed effects are amplitude, travel time and polarization anomalies and shifts in the location of spectral peaks. Several authors made use of these effects to invert surface wave data for a laterally heterogeneous earth model (e.g. Woodhouse & Dziewonski 1984; Tanimoto & Anderson 1985; Nataf, Nakanishi & Anderson 1986). The inversion methods are based on the great circle path approximation, in which deviations of the ray-paths from great circles are neglected. Observations indicate however, that appreciable departures from this approximation do occur on the Earth (Lay & Kanamori 1985; Roult, Romanowicz & Jobert 1986). The inversion methods are constrained to the great circle path approximation, in which deviations of the ray-paths from great circles are neglected. Observations indicate however, that appreciable departures from this approximation do occur on the Earth (Lay & Kanamori 1985; Roult, Romanowicz & Jobert 1986). Nevertheless, the laterally heterogeneous earth models give us the opportunity to study the propagation of surface waves on such models applying more sophisticated theories, and to check the validity of the great circle approximation.

Several efforts towards a more rigorous theory on the propagation of surface waves in laterally heterogeneous media have been made. Kennett (1984) derived an exact theory for surface wave propagation in two dimensions. Snieder (1986) devised a scattering theory for surface waves on 3-D media using the Born approximation. But the applicability of both methods is constrained to localized heterogeneities embedded in a smooth or laterally homogeneous background medium. An approach which is constrained to smooth variations of the lateral heterogeneities compared to the wavenumber considered, but is applicable to global propagation of surface waves, is ray theory. It was employed by Lay & Kanamori (1985) and Jobert & Jobert (1983) to show that the ray-paths may indeed significantly deviate from great circles. Woodhouse & Wong (1986) developed an inversion procedure for both phases and amplitudes, which is based on ray theory.

This paper deals with the application of the Gaussian beam method to surface waves. The Gaussian beam method is an extension of ray theory, which has been developed by Popov (1982) and Červený, Popov & Pšenčík (1982) for body waves. It has been the subject of an intensive discussion in the literature (Felsen 1984; Beydoun & Ben-Menahem, 1985; 1986).
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1985), because it suffers from a free constant not determined by theory. Until now, there is no rigorous way to fix this constant, although theoretical work on this subject is emerging (Einziger, Raz & Shapiro 1986). On the other hand, the Gaussian beam method has been thoroughly tested (Nowack & Aki 1984; Weber 1988), and is known to yield reliable results in the 2-D case even for complicated structures.

The Gaussian beam method was modified by Yomogida (1985) to treat the propagation of surface waves on a half space. Yomogida & Aki (1985) and Jobert (1986) applied this theory to surface waves propagating on a sphere using a Mercator transformation of the spherical coordinates and an associated transformation of the phase velocity function. But this procedure does not account properly for the curvature of the earth which becomes important for fundamental modes with periods greater than 100 s and also for higher modes because their eigenfunctions reach further down into the mantle. Since on a sphere, all displacements may be described by modes, the spherical approach yields a quite general description of elastic displacements within the Earth.

Incorporating the curvature of the Earth into the Gaussian beam method requires a ray-centred coordinate system which is adjusted to the spherical geometry. One objective of this paper is to provide a method for constructing a ray-centred coordinate system on an arbitrary smooth, curved surface. It will be shown that it is possible to derive a very simple dynamic ray-tracing system valid on such a surface. Specializing to a sphere, a spherical ray-centred coordinate system and its metric will be derived. A brief outline of the theory of Gaussian beams for surface waves on a sphere emphasizing the differences in derivation between the plane and the spherical case follows. It will be shown how the surface wave field on a laterally homogeneous Earth may be decomposed into Gaussian beams. Finally, we present some synthetic examples of anomalous surface wave propagation obtained with the Gaussian beam method and compare them with data measured at the station BFO.

2  RAY-CENTRED COORDINATES ON CURVED SURFACES

Consider a 2-D surface embedded in 3-D Euclidean space. Let \( x^2 \) and \( x^3 \) be coordinates on the surface while \( x^1 \) is reserved for an eventual third space coordinate defined along the normal of the surface. We use Greek indices for vector and tensor components on the surface. Summation is implied over all indices occurring twice in a product. Further we use a subscript behind a comma to denote differentiation with respect to the coordinate defined by the subscript.

Let \( g_{\alpha\beta} \) be the metric components and \( \Gamma^\nu_{\mu\nu} \) be the Christoffel symbols belonging to the coordinates \( x^\alpha \). We assume that on the surface a phase velocity function \( c(x^2, x^3) \) is given, allowing us to compute rays by extremizing the travel time

\[
\tau = \int_{s} \frac{1}{c} \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}} ds.
\]

(1)

\( s \) is the arclength along the ray-path.

The Lagrangian differential equations of (1) yield the second-order ray equations

\[
\frac{d^2x^\alpha}{ds^2} + \left( \Gamma^\alpha_{\mu\nu} + \frac{1}{c^2} g^{\mu\nu}(c_{,\mu}g_{\rho\nu} - c_{,\nu}g_{\rho\mu}) \right) \frac{dx^\rho}{ds} \frac{dx^\nu}{ds} = 0, \tag{2}
\]

which are equivalent to the general ray equations given by Woodhouse & Wong (1986). For a constant phase velocity, (2) reduces to the geodesic equation (Misner, Thorne & Wheeler 1973, p. 263) on the surface. The ray-paths coincide with the geodesics of the surface.

We define now a ray-centred coordinate system in the following way (Fig. 1): let \( P \) be a point on the surface in the vicinity of the ray. We draw a geodesic through \( P \) which intersects the ray at right angles at the point \( Q \). The ray-centred coordinate \( s \) is then the arclength \( OQ \), while the second coordinate \( n \) is given by the arclength \( PQ \). We define a ray-centred basis vector \( G_2 \) in such a way that \( G_2As \) is the separation vector between two neighbouring geodesics \( PQ \) and \( PO \). On the ray, \( G_2 \) coincides with the tangent vector of the ray. The second ray-centred basis vector, \( G_3 \), is defined as the unit tangent vector of the geodesic \( PQ \). The components \( G_3^\alpha \) satisfy the geodesic equation

\[
\frac{DG_3^\alpha}{Dn} = \Gamma^\alpha_{\nu\mu}G_3^nG_3^\nuG_3^\mu = 0. \tag{3}
\]

\( D/Dn \) denotes the covariant derivative with respect to \( n \).

Since \( G_2As \) is the separation vector between two neighbouring geodesics, the components \( G_3^\alpha \) satisfy the equation of geodesic deviation (Misner et al. 1973, p. 275)

\[
\frac{D^2G_3^\alpha}{Dn^2} + R^\alpha_{\nu\mu}G_3^nG_3^\nuG_3^\mu = 0, \tag{4}
\]

where \( R^\alpha_{\nu\mu} \) are the components of the Riemann curvature tensor of the surface.

In two dimensions, the Riemann curvature tensor may be expressed by a scalar, the Ricci curvature scalar, and the metric tensor (Misner et al. 1973, p. 334)

\[
R^\alpha_{\nu\mu} = \frac{1}{4}K(\delta^\alpha_{\nu}\delta_{\mu\nu} - \delta^\alpha_{\nu}\delta_{\mu\nu}). \tag{5}
\]

Inserting (5) into (4), we obtain

\[
\frac{D^2G_3^\alpha}{Dn^2} + \frac{1}{4}K(G_{33}G_3^\alpha - G_{32}G_3^\alpha) = 0, \tag{6}
\]

Figure 1. Definition of the ray-centred coordinates.
where $G_{\alpha\nu}$ are the metric components of the ray-centred coordinate system.

In principle, equations (3) and (6) allow us to compute all the components of $G_2$ and $G_3$, if initial conditions are given. But solving (3) and (6) may be quite a difficult task for general curved surfaces. It is much easier to determine the four metric components $G_{\alpha\nu}$.

Since $G_3$ is a unit vector, we have

$$G_{33} = 1. \quad (7)$$

Multiplying (6) by $g_{\alpha\beta} G_3^\beta$, we obtain

$$g_{\alpha\beta} G_3^\alpha \frac{D^2 G_3^\beta}{Dn^2} = 0. \quad (8)$$

Because the covariant derivatives of $g_{\alpha\beta}$ and $G_3^\beta$ both vanish, (8) may be written

$$\frac{d^2 (g_{\alpha\beta} G_3^\beta)}{dn^2} = \frac{d^2 G_3^\beta}{dn^2} = 0. \quad (9)$$

From (7) and the fact that

$$\frac{DG_3^\alpha}{Dn} = \frac{DG_3^\alpha}{D\gamma}, \quad (10)$$

it can be shown that (9) also implies

$$G_{23} = 0. \quad (11)$$

(11) can also be derived if $G_3$ is not chosen as a unit vector. In this case, a sufficient condition for (11) to be valid is that $G_{23}$ does not depend on $s$. We conclude, that the ray-centred coordinate system, defined in the described way, has a diagonal metric tensor for all curved surfaces. It is always possible to define an orthogonal ray-centred coordinate system on a curved surface.

Multiplying (6) by $g_{\alpha\beta} G_2^\beta$ and differentiating with respect to $n$ using (7) and (11) yields a third order differential equation for $G_{22}$:

$$\frac{d^3 G_{22}}{dn^3} + 2K \frac{dG_{22}}{dn} + \frac{dK}{dn} G_{22} = 0, \quad (12)$$

to be solved under the initial conditions

$$G_{22}(0) = 1, \quad \left. \frac{dG_{22}}{dn} \right|_0 = \frac{cG_{22}}{c}, \quad \left. \frac{d^2 G_{22}}{dn^2} \right|_0 = 2 \left( \frac{cG_{22}}{c} \right)^2 - K(0). \quad (13)$$

The first initial condition follows from the fact that on the ray $G_2$ coincides with the unit tangent vector of the ray. The second one can be derived if the ray equations (2) are evaluated in ray-centred coordinates. The third one can be derived by multiplying (6) by $g_{\alpha\beta} G_2^\beta$ and evaluating this expression on the ray in ray-centred coordinates using (10). With (7), (11), (12) and (13) the metric tensor of the ray-centred coordinate system is determined. It depends only on the curvature of the ray, $\epsilon_n/c$, and the Ricci curvature scalar of the surface, which in two dimensions is just twice the Gaussian curvature of the surface.

**3 GAUSSIAN BEAMS FOR SURFACE WAVES ON A SPHERE**

In this section we give a brief outline of the derivation of Gaussian beams for surface waves on a spherical earth. We assume arbitrary radial heterogeneities but admit only slight lateral heterogeneities. In addition, it is assumed that the wavelength considered is small against a characteristic length of the lateral heterogeneities. For simplicity, we assume an earth without internal boundaries. The elastodynamic boundary value problem

$$\rho \frac{\partial^2 u}{\partial t^2} = \text{div} \mathbf{T}, \quad \mathbf{f} \cdot \mathbf{T}_{\mid r=0} = 0, \quad (14)$$

is solved in ray-centred coordinates. $\mathbf{T}$ is the stress tensor, $\rho$ density, $\mathbf{u}$ displacement, $\mathbf{f}$ is the unit vector in radial direction and $a$ is the radius of the Earth.

Consider a point on a spherical shell of radius $r$ with coordinates $s$ and $n$ as defined in Fig. 1. On this shell, the Ricci curvature scalar is given by (Misner et al. 1973, p. 341)

$$K = \frac{2}{r^2} \quad (15)$$

and (12) with the initial conditions (13) may be easily solved. We obtain

$$G_{22} = \left( \frac{\cos \gamma + \frac{cG_{22}}{r} \sin \gamma}{c} \right)^2, \quad G_{23} = r^2. \quad (16)$$

Taking the limit $r \to \infty$, we recover from (16) and (7) the metric of the ray-centred coordinate system for a plane surface as it may be found in Yomogida (1985). Unfortunately, the coordinates $s$ and $n$ are not well suited for our problem because they depend on the radius of the shell. Therefore, we define radius-independent angular coordinates by

$$ds = r^{-1} \, ds \quad \text{and} \quad d\gamma = r^{-1} \, d\gamma. \quad (17)$$

Analogously, the phase velocity at given latitude and longitude depends linearly on the radius. To avoid this $r$-dependence, we introduce an angular phase velocity

$$\mathbf{c}(\sigma, \gamma) = (\mathbf{c}(r, \sigma, \gamma))/r, \quad (18)$$

and obtain instead of (16) and (7)

$$G_{\alpha\beta} = r^2 \left( \cos \gamma + \frac{cG_{22}}{c} \sin \gamma \right)^2, \quad G_{22} = r^2. \quad (19)$$

We try to solve (14) with the ansatz of a surface wave travelling with angular phase velocity $\mathbf{c}(\sigma)$ along a given ray:

$$\mathbf{u}(r, \sigma, \gamma) = \left( \begin{array}{c} U_r(r, \sigma, \gamma) \\ iU_\varphi(r, \sigma, \gamma) \end{array} \right) \exp \left( i \sigma \int \frac{d\sigma}{\mathbf{c}(\sigma)} - i\tau \right) \quad (20)$$

and expand $\mathbf{U}$ into a perturbation series

$$\mathbf{U} = \mathbf{U}^{(0)} + \omega^{-1/2} \mathbf{U}^{(1)} + \omega^{-1} \mathbf{U}^{(2)} + \cdots \quad (21)$$

with $\omega^{-1/2}$ as perturbation parameter (Červený et al. 1982).

Evaluating (14) in the metric (19) and inserting (20) with (21) yields an equation which can be ordered with respect to powers of $\omega^{-1/2}$:

$$\sum_{k=0}^{\infty} \omega^{2-k/2} (L_2^{(k)} + L_3^{(k)} + L_4^{(k)}) = 0, \quad (22)$$

$$\sum_{k=0}^{\infty} \omega^{1-k/2} (B_2^{(k)} + B_3^{(k)} + B_4^{(k)}) = 0. \quad (23)$$
The \( L_n \) are dyadic differential operators, while the \( B_n \) are dyadic boundary operators acting on \( \mathbf{U} \) at the surface \( r = a \). To achieve the proper ordering of all terms, the coordinates \( r \) and \( \gamma \) had to be rescaled (Cerveny et al. 1982):

\[
R = \omega r, \quad \gamma = \frac{\omega}{c} \gamma.
\]  

(24)

Since explicit expressions of the operators \( L_n \) are very spacious, we give here only the zeroth order operators with the following abbreviations:

\[
\omega^2 L_0 \mathbf{U}^{(k)} = \left\{ \begin{array}{l}
\frac{\partial}{\partial r} \left( r \lambda + \frac{\lambda}{r} \right) \mathbf{U}^{(k)} + \lambda \omega^2 \mathbf{U}^{(k)} - \frac{\omega}{h \varepsilon} L_{12} \mathbf{U}^{(k)}, \\
\frac{1}{r^2} \frac{\partial}{\partial \gamma} \left( r^2 \lambda + \frac{\lambda}{r^2} \right) \mathbf{U}^{(k)}, \\
\frac{\partial}{\partial \gamma} \left( r^2 \lambda + \frac{\lambda}{r^2} \right) \mathbf{U}^{(k)}, \\
\left( 2 \lambda + \mu \right) \frac{\partial}{\partial \gamma} \mathbf{U}^{(k)} - \frac{\omega \lambda}{h \varepsilon} \mathbf{U}^{(k)}, \\
\left( \frac{\partial}{\partial \gamma} - 2 \right) \mathbf{U}^{(k)},
\end{array} \right.
\]  

(25)

\[
\omega^2 B_1 \mathbf{U}^{(k)} = \left[ \begin{array}{l}
\frac{\mu \omega^2}{h \varepsilon} \mathbf{U}_r + \mu \frac{\partial}{\partial \gamma} \mathbf{U}^{(k)}, \\
\mu \frac{\partial}{\partial \gamma} \mathbf{U}^{(k)}, \\
\left( \frac{\partial}{\partial \gamma} - 1 \right) \mathbf{U}^{(k)},
\end{array} \right.
\]  

(26)

with the following abbreviations:

\[
\begin{align*}
L_{12} &= -\frac{3 \lambda + \mu}{r} + (2 \lambda + \mu) \frac{\partial}{\partial \gamma}, \\
L_{21} &= -\lambda + (2 \lambda + \mu) \frac{\partial}{\partial \gamma}.
\end{align*}
\]  

(27)

Further, we wrote \( h \) instead of \( \sqrt{G_{00}} \). Explicit expressions for the operators \( L_{3/2}, L_1, B_{1/2} \) and \( B_0 \) may be found in Friederich (1988).

From now on, we distinguish between Rayleigh and Love waves. We treat here only Rayleigh waves, since the derivation for Love waves follows the same lines. For Rayleigh waves, the phase velocity \( \varepsilon(\sigma) \) in (20) is equal to the angular phase velocity of the Rayleigh wave \( \varepsilon_R(\sigma, \gamma) \) on the ray, i.e.

\[
\varepsilon(\sigma) = \varepsilon_R(\sigma, 0).
\]  

(28)

Generalizing the approach of Yomogida (1985), we assume that the \( k \)th order displacement \( \mathbf{U}^{(k)} \) can be decomposed into two vector fields

\[
\mathbf{U}^{(k)} = F^{(k)}(\sigma, \gamma) s_0 + \mathbf{P}^{(k)},
\]  

(29)

where \( s_0 \) is the local spheroidal eigenfunction and \( \mathbf{P}^{(k)} \) is a vector field still to be determined. \( s_0 \) is normalized in the following way:

\[
\int_0^a \rho s_0^2 r^2 \, dr = 1.
\]

Note that \( s_0 \) is the local eigenfunction of a laterally heterogeneous sphere and not a half-space. The local eigenfunction \( s_0 \) obeys the differential equation

\[
\omega^2 L_0 s_0 = 0 \quad \text{with} \quad \omega B_0 s_0 = 0.
\]  

(30)

The operators \( L_0 \) and \( B_0 \) have the same form as the operators \( L_2 \) and \( B_1 \) of (25) and (26), except that the term \( h \varepsilon \) is replaced by \( r \varepsilon_R \). By a straightforward partial integration, it can be shown that Green's formula is valid for \( L_0 \). Here we state it using the boundary operator \( B_0 \):

\[
\omega^2 \int_0^a \left( \mathbf{v} \cdot \mathbf{L} \mathbf{u} - \mathbf{u} \cdot \mathbf{L} \mathbf{v} \right) r^2 \, dr = \omega^2 [\mathbf{v}(a) \cdot \mathbf{B}_0 \mathbf{u} - \mathbf{u}(a) \cdot \mathbf{B}_0 \mathbf{v}].
\]  

(31)

Since \( L_0 \) contains only derivatives with respect to \( r \), \( F^{(k)} s_0 \) is an eigenfunction of \( L_0 \). Therefore, we may write

\[
\omega^2 L_2 \mathbf{U}^{(k)} = \omega^2 (L_0 - L_2) F^{(k)} s_0 + \omega^2 L_2 \mathbf{P}^{(k)}
\]

\[
\omega B_1 \mathbf{U}^{(k)} = \omega (B_0 - B_1) F^{(k)} s_0 + \omega B_1 \mathbf{P}^{(k)}.
\]

(32)

The operator differences in (32) consist mainly of expressions like \( (h \varepsilon)^{-2} - (r \varepsilon_R)^{-2} \) and \( (h \varepsilon)^{-1} - (r \varepsilon_R)^{-1} \) which vanish on the ray since there \( \varepsilon_R \) is equal to \( \varepsilon_R \) and \( h = \varepsilon_R \). We expand both expressions into a Taylor series at \( y = 0 \):

\[
\begin{align*}
\frac{1}{h \varepsilon^2} - \frac{1}{r \varepsilon_R^2} &= \frac{1}{r \varepsilon_R^2} \sum_{n=2}^\infty \omega^{-n/2} \varphi^n T_n^{(2)}, \\
\frac{1}{h \varepsilon} - \frac{1}{r \varepsilon_R} &= \frac{1}{r \varepsilon_R} \sum_{n=2}^\infty \omega^{-n/2} \varphi^n T_n^{(1)}.
\end{align*}
\]

Here \( T_n^{(1)} \) is the \( n \)th derivative of the corresponding left-hand side with respect to \( \gamma \) on the ray divided by \( n! \) and multiplied by \( \varphi^n \). For example,

\[
T_2^{(2)} = \frac{\xi^2 + 1}{\varepsilon^2}.
\]

(34)

This expression is different from the corresponding expression in a plane geometry, where the \( 1/\varepsilon^2 \) term is missing. The reason is that the metric component \( G_{00} \) and therefore the scale factor \( h \) in (33) depend on the geometry of the surface. Defining the matrices \( \mathbf{E}_n \) and \( \mathbf{D}_n \) by

\[
\begin{pmatrix}
\frac{\mu \omega^2}{r c^2} T_n^{(2)} & -\omega \frac{L_{12}}{r c} T_n^{(1)} & 0 \\
\omega \frac{L_{21}}{r c} T_n^{(1)} & -\lambda + \mu \omega^2 T_n^{(2)} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

(35)

\[
\omega \mathbf{D}_n = \begin{pmatrix}
0 & -\omega \lambda & 0 \\
\frac{\mu}{r c} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} T_n^{(1)}
\]

and approximating the values of the local eigenfunction and the elastic parameters by their values on the ray, we obtain

\[
L_2 \mathbf{U}^{(k)} = L_0 \mathbf{U}^{(k)} + \sum_{n=2}^{\infty} \omega^{-n/2} \gamma^n \mathbf{E}_n F^{(k)} s_0
\]

\[
B_1 \mathbf{U}^{(k)} = B_0 \mathbf{U}^{(k)} + \sum_{n=2}^{\infty} \omega^{-n/2} \gamma^n \mathbf{D}_n F^{(k)} s_0.
\]

(36)

Inserting (36) into (22) and (23), we obtain for the equation
Thus, we have an inhomogeneous differential equation with form

\[ \omega^2 - k^2 \left( L_u U^{(k)} + L_{3/2} U^{(k-1)} + L_1 U^{(k-2)} + L_{1/2} U^{(k-3)} \right) \]

+ \[ L_0 U^{(k-4)} + \sum_{n=2}^{k} \gamma^n E_n F^{(k-n)} s_0 = 0 \]

\[ \omega^1 - k^2 \left( B_{3/2} U^{(k)} + B_{3/2} U^{(k-1)} + B_1 U^{(k-2)} + B_{1/2} U^{(k-3)} \right) \]

+ \[ B_0 U^{(k-4)} + \sum_{n=2}^{k} \gamma^n D_n F^{(k-n)} s_0 = 0. \] (37)

The ansatz (29) does not contradict (37). To see this, assume that \( P^A(k) \) of (29) is a particular solution of (37). Since \( s_0 \) is an eigenfunction of \( L_s \), we may always add \( s_0 \) multiplied by a function which does not depend on \( r \) to the particular solution. Setting \( k = 0 \) in (37) yields the zeroth order equation

\[ \omega^2 L_u U^{(0)} = 0 \quad \text{and} \quad \omega B_s U^{(0)} = 0. \] (38)

They are satisfied if

\[ U^{(0)} = 0 \quad \text{and} \quad P^{(0)} = 0. \] (39)

From the first-order equations we obtain (Friederich 1988)

\[ P^{(1)} = 0, \quad U^{(1)} = \varepsilon U_{\sigma, \gamma}. \] (40)

\( U^{(1)} \) is an additional component due to the curvature of the wavefront. Putting \( k = 2 \) in (37) and using (39) with (29), we get

\[ L_u U^{(2)} = -L_{3/2} U^{(1)} - (L_1 + \gamma^2 E_2) F^{(0)} s_0 \]

\[ B_s U^{(2)} = -B_{3/2} U^{(1)} - (B_1 + \gamma^2 D_2) F^{(0)} s_0. \] (41)

Substituting \( U^{(1)} \) by virtue of (40), we may write (41) in the form

\[ L_u U^{(2)} = -f \]

\[ B_s U^{(2)} = -g. \] (42)

Thus, we have an inhomogeneous differential equation with inhomogeneous boundary conditions for \( U^{(2)} \). The right-hand sides of (42) depend on \( F^{(0)} \) which is not determined yet. However, we cannot solve (42) for arbitrary values of its right-hand sides and therefore of \( F^{(0)} \), simply because \( s_0 \) is an eigenfunction of \( L_s \). This fact may be expressed in a necessary condition for the existence of \( U^{(2)} \), which follows directly from (31), if we choose \( u = s_0 \) and \( v = U^{(2)} \) and apply (42):

\[ \omega^2 r^2 f \cdot s_0 dr = \omega a^2 s_0(a) \cdot g. \] (43)

(43) allows us to compute the unknown function \( F^{(0)} \) and therefore \( U^{(0)} \). If \( f \) and \( g \) are explicitly inserted into (43) a parabolic equation for \( F^{(0)} \) is obtained. Thus, for surface waves, the parabolic equation yielding the zeroth order displacement \( U^{(0)} \) follows directly from the solvability condition imposed on the inhomogeneous part of (42). A solution of (43) is a Gaussian beam of the form

\[ \left( \begin{array}{c} U^{(0)} \\ U^{(0)} \end{array} \right) = H \sqrt{q(0) G(0)} \begin{pmatrix} U(r, \sigma) \\ V(r, \sigma) \end{pmatrix} \exp \left( \frac{i p(\sigma)}{2 q(\sigma)} \gamma^2 \right). \] (44)

where \( q \) and \( p \) are generally complex and may be computed from the dynamic ray-tracing system

\[ \frac{dq}{d\sigma} = \varepsilon p \quad \text{and} \quad \frac{dp}{d\sigma} = -\left( \frac{\gamma^2}{\varepsilon^2} + \frac{1}{c} \right) q. \] (45)

A detailed derivation of (44) is given in Friederich (1988). \( H \) is a free constant which is determined by the source.

Note, that in the second equation of the dynamic ray-tracing system an additional \( 1/\varepsilon \) term turns up, which is not present in the plane case. This term cares for the refocusing of the rays at the antipodes. (45) allows us to compute amplitudes of Gaussian beams on a sphere for any source receiver pair, even if the source is located at one of the poles. This fact greatly alleviates the evaluation of the constant \( H \) in the next section. (45) has two fundamental solutions \( (q_1, p_1) \) and \( (q_2, p_2) \) from which the general solution can be constructed by superposition:

\[ q(\sigma) = q_1(\sigma) + \delta q_2(\sigma) \]

\[ p(\sigma) = p_1(\sigma) + \delta p_2(\sigma). \] (46)

The constant \( \delta \) is generally complex and is known as the beam constant. A complex beam constant guarantees that \( q \) never vanishes and that the displacement is always regular.

In the following, we shall show that the dynamic ray-tracing system (45) can also be derived as a special case using the general formalism of Section 2. This is due to the fact that (45) describes a paraxial ray if \( q \) and \( p \) are real. Let the paraxial ray be given by \( u^p + h^p \), where \( h^p \) is assumed to be small. Perturbing the ray equations (2), we obtain

\[ \frac{d^2 h^p}{ds^2} + \frac{H_{p \nu}^a}{H_{p \nu}^a} \left( \left( \frac{ds}{ds} \right)^2 + \left( \frac{ds}{ds} \right)^2 \right) + H_{p \nu}^a h^p \frac{dx^p}{ds} \frac{dx^\nu}{ds} = 0, \] (47)

with

\[ H_{p \nu}^a = \Gamma_{p \nu}^a + \frac{1}{c} \delta_{p \nu} \varepsilon_{\mu \rho \sigma}(c_{\mu \rho} \delta_{p \sigma} - c_{\mu \rho} \delta_{p \sigma}). \]

(47) is equivalent to equation (18) of Woodhouse & Wong (1986), which was derived using a Hamiltonian formalism. (47) may be quite difficult to solve depending on the coordinate system chosen. However, in general ray-centred coordinates, (47) takes a very simple form and even decouples partially. Calculating the Christoffel symbols from the metric components (19) using (13), we obtain

\[ \frac{d^2 h^p}{ds^2} + \frac{d}{ds} \left( \frac{1}{c} \frac{dh^p}{ds} \right) + \left( \frac{c_{\mu \nu}}{c^2} + \frac{K}{2c} \right) \frac{1}{c} h^p = 0, \] (48)

where

\[ c_{\mu \nu} = G_{\mu}^a G_{\nu}^b \left( c_{\mu \beta} - c_{\mu} \Gamma_{\nu}^\beta \right). \]

It should be noted that the use of the general ray centred coordinates leads also to a partial decoupling of equation (25) and equation (28) of Woodhouse & Wong (1986). The required Green’s functions are easily obtained from the solutions of the second equation of (48).

The second equation of (48) is the appropriate dynamic ray-tracing system for a curved surface. Note that the geometry of the surface enters only by its intrinsic curvature \( K/2 \). We may recover (45) from the general dynamic...
ray-tracing system by making the substitutions
\[ q = \frac{h^3}{a} \text{ and } p = \frac{a dh^3}{c \, ds} \]
and using (15), (17) and (18).

We may also conclude from the general definition of the ray-centred coordinates in Fig. 1, that \( q \) is the distance between the central and a paraxial ray measured along a great circle perpendicular to the central ray.

Finally, taking together (44), (40), (21) and (20), we may write the Fourier transform of the displacement of a Rayleigh wave as follows:

\[
\mathbf{u}_r(r, \sigma, \gamma, \omega) = H_R \sqrt{\frac{q(0) \tilde{G}(0)}{q(\sigma) \tilde{G}(\sigma)}} \times \left[ U(r, \sigma) \hat{r} + i \left( \tilde{\mathbf{G}}_\sigma + \frac{\gamma c \mathbf{P}}{q} \tilde{\mathbf{G}}_\gamma \right) V(r, \sigma) \right] \times \exp \left[ i \omega \left( \frac{da}{c(\sigma)} + \frac{1}{2} \frac{p}{q} \gamma^2 \right) \right].
\]

For Love waves, following similar steps, we obtain:

\[
\mathbf{u}_l(r, \sigma, \gamma, \omega) = H_L \sqrt{\frac{q(0) \tilde{G}(0)}{q(\sigma) \tilde{G}(\sigma)}} \left( \tilde{\mathbf{G}}_\sigma - \frac{\gamma c \mathbf{P}}{q} \tilde{\mathbf{G}}_\gamma \right) W(r, \sigma) \times \exp \left[ i \omega \left( \frac{da}{c(\sigma)} + \frac{1}{2} \frac{p}{q} \gamma^2 \right) \right].
\]

Here, \( \tilde{\mathbf{G}}_\sigma \) and \( \tilde{\mathbf{G}}_\gamma \) are the normalized basis vectors of the ray-centred coordinate system. They can be obtained from the basis vectors \( \mathbf{G}_\sigma \) and \( \mathbf{G}_\gamma \) of Section 2 by multiplying with \( r \) and dividing by the square root of the corresponding metric component given in (19).

4 SUPERPOSITION OF GAUSSIAN BEAMS

The beams (50) and (49) give us the displacement in the vicinity of a ray. If we want to compute the displacement at a receiver generated by a point source, we have to integrate over all beams emanating from the source. In practice, the integration is discretized into a summation over a finite number of beams. Since, with a complex beam constant, the displacement has a Gaussian amplitude profile normal to the ray, it depends on the width of the Gaussian how many beams contribute at the receiver. To carry out the superposition of the beams properly, the constant \( H \) has to be determined for all initial directions of the contributing beams. In principle, we should determine \( H \) in a laterally heterogeneous medium, but no analytical expressions for \( H \) are available in such a medium. Therefore, \( H \) is calculated in a laterally homogeneous medium, where an analytical expression of \( H \) as a function of the radiation angle can be derived. We consider in the following a laterally homogeneous Earth with a point source located at the pole and a receiver at the point \( P \) with coordinates \( \Delta \) and \( \phi \) (Fig. 2). Our aim is to compute the displacement of the first wavetrain at \( P \). It is represented by superposition of Gaussian beams emanating from the pole with initial direction \( \lambda \):

\[
\mathbf{u}_{L,R}(r, \Delta, \phi, \omega) = \int_{\phi-\pi}^{\phi+\pi} d\lambda \mathbf{u}_{\text{Beam}, L,R}(r, \sigma, \gamma, \omega).
\]

It follows from Fig. 2, that \( \sigma \) and \( \gamma \) depend on \( \lambda \). The same is true for the ray-centred basis vectors occurring in (49) and (50). To avoid this \( \lambda \)-dependence, we first transform the ray-centred basis vectors to the spherical basis vectors at \( P \). This transformation is achieved by just rotating the ray-centred basis vectors by an angle \( \alpha - \pi/2 \) (see Fig. 2). Thus we may write

\[
\begin{align*}
\hat{r}(\gamma) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \\
\hat{\mathbf{e}}_\sigma(\gamma) &= \begin{pmatrix} 0 & \sin \alpha & -\cos \alpha \end{pmatrix}, \\
\hat{\mathbf{e}}_\gamma(\gamma) &= \begin{pmatrix} 0 & \cos \alpha & \sin \alpha \end{pmatrix},
\end{align*}
\]

where \( \hat{r} \), \( \mathbf{A} \) and \( \Phi \) are the unit basis vectors of the spherical coordinate system. On a laterally homogeneous Earth, the phase velocity is constant and we may write the phase of the beam as

\[
\frac{\omega}{c} \left( \sigma + \frac{c \mathbf{P}}{2 q} \gamma^2 \right),
\]

where \( p \) and \( q \) can be expressed from (45) by

\[
q(\sigma) = \cos \sigma + \delta \sin \sigma
\]

\[
p(\sigma) = \frac{1}{c} (-\sin \sigma + \delta \cos \sigma).
\]

An approximate expression for the integral (51) can be obtained, if the method of stationary phase is applied. To this purpose, the phase term (53) is expanded up to the second order around \( \lambda = \phi \) yielding

\[
\frac{\omega}{c} \left( \sigma + \frac{c \mathbf{P}}{2 q} \gamma^2 \right) = \frac{\omega}{c} \left( \Delta - \frac{1}{2 \cos \Delta + \delta \sin \Delta} (\lambda - \phi)^2 \right),
\]

while all other terms are evaluated at \( \lambda = \phi \), implying
$\alpha = \pi/2$ and $\gamma = 0$. Since the integrand decays according to a Gaussian, we may extend the integration limits to infinity and obtain from (51)

$$\mathbf{u}_L = \left( \frac{2\pi c}{\omega \sin \Delta} \right)^{1/2} \exp \left( \frac{i\omega c}{\epsilon} \Delta \right) H_\ell(\phi) W(r) \Phi,$$

$$\mathbf{u}_R = \frac{i\pi c}{2 \pi \omega \sin \Delta} \frac{1}{\epsilon} \exp \left( \frac{i\omega c}{\epsilon} \Delta + \frac{i\pi}{4} \right) \left[ \frac{i}{G} \right] U(r) \Phi.$$

On a laterally homogeneous earth, it is possible to represent the far-field displacement as a sum of surface wavetrains travelling around the Earth along the great circle which leads through source and receiver. This representation follows from the usual normal mode representation by the application of the Poisson sum formula and an asymptotic expression for the spherical harmonics involved in the normal mode sum (Aki & Richards 1980). The displacement of the first wavetrain of a Rayleigh wave can be written

$$\mathbf{u}_L = \frac{i\pi c}{2 \pi \omega \sin \Delta} \frac{1}{\epsilon} \exp \left( \frac{i\omega c}{\epsilon} \Delta + \frac{i\pi}{4} \right) \left[ \frac{i}{G} \right] U(r) \Phi.$$

A comparison of (55) and (56) yields

$$H_{L, \text{odd}}(\lambda) = \sum_{m=-2}^{2} b_m N_m \left( \frac{\omega}{c} \right)^{|m|} \exp \left[ \frac{im(\phi - \pi/2)}{2G} \right].$$

Here we used

$$b_m = \frac{1}{\omega^2} \sum_{p,q} e_{pq} M_{pq}(\omega),$$

where $e_{pq}$ are the strain components at the source depth belonging to the eigenfunctions $U, V, W$ and $M_{pq}$ are the components of the Fourier transform of the moment rate tensor. $N_m$ are the normalization constants of the spherical harmonics.

Since the weight functions $H_{L,R}$ depend only on the source and the initial direction of the ray, they are also valid for all higher odd wavetrains. Following a very similar derivation, one obtains for the even wavetrains

$$H_{L, \text{even}}(\lambda) = \sum_{m=-2}^{2} b_m N_m \left( \frac{\omega}{c} \right)^{|m|} \exp \left[ \frac{im(\lambda + \pi/2)}{2G} \right].$$

The plus sign in the exponential term comes from the different initial direction of the rays of the even wavetrains.

## 5 SYNTHETIC SEISMOGRAMS

We computed synthetic seismograms using the laterally heterogeneous mantle model M84C of Woodhouse & Dziewonski (1984). Each seismogram contains the 71 fundamental modes from $\nu S_{20}$ to $\nu S_{90}$, corresponding to a period range from 350 to 100 s. For each mode and

Figure 3. Ray-paths for a source near the Kuriles and a receiver on Iceland for modes $\nu S_{45}$ and $\nu S_{65}$ from bottom to top. The world maps are repeated to prevent an overlapping of the rays. The source is located in the third map from the left. The receiver is marked by a star.
wavetrain, 11 beams, encompassing an angle of 60° at the source, were superimposed. To compute the weight functions (57) and (58), we used the centroid moment tensor solutions of Dziewonski et al. (1983a,b, 1985). The beam constant $b$, (equation 46), was chosen in such a way that the integral over the square of the beam width taken along the ray is minimized, an option, which stems originally from Klímaš and is described by Weber (1988). Our experience has shown that this way of choosing the beam constant leads to very stable beam widths, nearly independent of the initial direction of the ray. A comparison of Gaussian beam seismograms, obtained with this option, with an exact solution for a laterally homogeneous earth model yielded very good agreement for epicentral distances greater than 40° (Friederich 1988). Near the antipode of the source we got very unsatisfactory results both for a laterally homogeneous and a laterally heterogeneous earth model. Obviously, there is considerable energy arriving at a receiver near the antipode of the source which does not propagate within the ray tube defined by the two outermost

Figure 4. Synthetic seismograms for a source near the Kuriles and a receiver on Iceland. Dashed line: Z-component for the laterally homogeneous reference model. Solid line: Z-component for model M84C. On the left-hand side the odd wavetrains are depicted starting with $R_1$ at the top, on the right-hand side the even wavetrains are depicted starting with $R_2$ at the top.
rays of a bundle. The Gaussian beam method appears not to be an appropriate instrument to model the surface wavefield in the vicinity of the source and its antipode.

The first synthetic example we present is the ray-paths and synthetic seismograms for a source near the Kuriles and a receiver on Iceland. In Fig. 3, the paths of six rays are depicted for modes $s_{05}$ and $s_{055}$. The receiver is marked by a star. There are quite large deviations from great circles which increase with higher orbit number. For the even wavetrains, there is a clear defocusing of the rays away from the mid-Atlantic ridge, while for the odd wavetrains the rays arrive at the station asymmetrically with respect to the great circle and the defocusing is not so distinct. We computed three different synthetic seismograms for this source–receiver pair: one for a laterally homogeneous earth model as a reference, one for model M84C and one for M84C three times exaggerated (M84C*3). In Fig. 4, the Gaussian beam seismogram for model M84C is plotted together with the reference seismogram. As it was to be expected from the ray-paths, there is a slight amplitude decrease for all wavetrains of higher order than $R_1$ together with a phase shift which increases with orbit number. The amplitude loss

![Image of Figure 5 showing synthetic seismograms for a source near the Kuriles and a receiver on Iceland. Dashed line: Z-component for the laterally homogeneous reference model. Solid line: Z-component for model M84C*3. Ordering of the wavetrains same as in Fig. 4.](http://gji.oxfordjournals.org/gji.oxfordjournals.org/)
is much more drastic in the Gaussian beam seismogram for M84C*3 (Fig. 5), where a significant part of the energy of the higher order wavetrains is deflected from the great circle connecting source and receiver. In contrast to Fig. 4, we have now phase shifts depending strongly on frequency. \( R_n \) is split apart by an amplitude minimum which may be explained by destructive interference of beams arriving at the receiver from the west and the east of the mid-Atlantic ridge.

In Figs 6–8 we show comparisons of Gaussian beam seismograms with data from an event south of Madagascar (MAD) and an event near the Prince Edward Islands (PEI) recorded both at BFO. The data were instrument corrected and reduced in spectral bandwidth to that of the synthetics. In Fig. 6 all four seismograms are depicted in one plot. The two at the top belong to PEI, those at the bottom to MAD. All four seismograms have been normalized to the same double amplitude of \( R_1 \) to facilitate a comparison of the amplitudes of the higher order wavetrains. The record of the PEI event contains considerable higher mode energy which may prevent a correct estimate especially of the amplitudes of wavetrains \( R_3 \) and to a less extent of \( R_4 \), for which the anisotropic earth model PREM of Dziewonski & Anderson (1981) predicts a partial overlap of the group travel-time curves of the fundamental mode and the second spheroidal overtone. In the MAD record the higher modes are much weaker and should therefore not considerably disturb the amplitude of the wavetrains. Both events differ in azimuth only by 10° and in epicentral distance by 2°. This fact explains why the higher order wavetrains of the synthetic seismograms in Fig. 6 have nearly identical amplitude ratios. The data, however, show despite of the vicinity of the travel paths significantly different amplitudes, especially at \( R_2, R_4 \), and \( R_5 \). Obviously, the differences in the data could only be modelled in the presence of lateral heterogeneities varying on a smaller scale than those contained in model M84C.

In Figs 7 and 8 synthetic seismograms and data of both events are depicted in one plot, respectively. Although the phases of the odd wavetrains are matched quite well for both events, there are significant phase shifts increasing with orbit number at the odd wavetrains. The amplitude misfit is largest at the even wavetrains for PEI (Fig. 7), while for MAD (Fig. 8) the largest amplitude discrepancy is at \( R_3 \). \( R_e \) of MAD has unfortunately not been recorded. In the case of MAD, it can be seen that the amplitude misfit may be quite large although the phase is matched quite well. One might argue that the Gaussian beam method yields erroneous results for wavetrains of higher order than \( R_3 \), because then scattered energy arriving sideways becomes important. To this argument we have the following objection: using Born scattering theory Snieder (1987) showed that for models with smooth heterogeneities the superposition of the scattered waves at the receiver results in a phase shift, a deflection of polarization and a perturbation of amplitude due to focusing. All of these effects are accounted for in ray theory. Thus, at least for smooth models Born scattering theory and ray theory are equivalent. We are therefore convinced that our synthetic seismograms contain the major part of the scattered energy. We conclude that the large amplitude misfits (e.g. \( R_5 \) in Fig. 8) are not artifacts created.

Figure 6. Comparison of synthetic and recorded seismograms: from top to bottom: recording of the Z-component of the Wielandt–Streckeisen seismometer at BFO of an event near the Prince Edward Islands; synthetic Z-component for the event at the Prince Edward Islands recorded at BFO; recording of the Z-component of the Wielandt–Streckeisen seismometer at BFO of an event south of Madagascar; synthetic Z-component for the event south of Madagascar recorded at BFO. All seismograms have identical double amplitude at \( R_1 \).
by approximations in theory but contain valuable information about Earth's structure. This conclusion is supported by the significantly different amplitudes in the MAD and PEI records, which indicate the existence of small-scale heterogeneities not included in M84C.

6 CONCLUSIONS
We described a general way to construct a ray-centred coordinate system on a curved surface and derived a third-order differential equation for one of its metric components together with the appropriate initial conditions. Applying this method to a spherical shell, we obtained a ray-centred coordinate system adjusted to the spherical geometry, which allowed us to derive the Gaussian beam method for surface waves on a sphere taking into account properly the curvature of the Earth. The well-known parabolic equation (Červený et al. 1982) for the zeroth order displacement turned out to be a necessary condition for the solvability of equation (42). A dynamic ray-tracing system
for curved surfaces was derived which is influenced by the geometry only through the intrinsic curvature of the surface. Furthermore, weight functions were derived allowing a synthesis of the surface wavefield on a sphere by superposition of Gaussian beams.

Computations of rays and synthetic seismograms for the laterally heterogeneous earth model M84C show that considerable deviations of the ray-paths from great circles can occur, producing focusing and defocusing resulting in amplitude anomalies which can be seen on the seismograms.

Comparison with data recorded at BFO shows that the fit of the synthetics to the data can be very poor, although the phases may be fitted quite well. One reason appears to be that the heterogeneities in the model are still too smooth.

**ACKNOWLEDGMENTS**

The author is greatly indebted to W. Zürn for initiating and supporting this research, and to D. Gajewski for many fruitful discussions. He wishes also to thank E. Wielandt for
providing the data from his seismographs at BFO. H. Schenk and E. Wielandt directed the author's attention to the anomalous travel path of Fig. 3.

This work was financially supported by the German Research Association (DFG) under grant number SFB 108-B5.

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