Characteristic correlation time as estimate of optimum filter length in Maximum Entropy Spectral Analysis

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Summary. An original conceptually simple criterion for the choice of filter length $M$ for Maximum Entropy Spectral Analysis based on the correlation time-scale of data set is proposed. The following formula is obtained from the autocovariance estimates $\phi_t$

$$M = \frac{\pi}{2} \frac{\sum |\phi_t|}{\phi_0}.$$

The application of the proposed criterion to various digital geophysical records has provided satisfactory results. The method is well suited to geophysical signals because they often have a remarkable Markovian character in the broader sense, i.e. the autocovariances have values decreasing roughly exponentially as the time lags increase. Computer simulations on mathematically defined time series confirm the reliability of this criterion.

1 Introduction

The Burg spectral estimator, or maximum entropy spectral density, is given by

$$P(f) = \frac{P_{M+1} \Delta t}{\left| 1 - \sum_{k=1}^{M} a_k \exp(-2\pi i k \Delta t) \right|^2}$$

where $f$ is the frequency, $\Delta t$ the sampling rate, $P_{M+1}$ the output power of the prediction error filter, whose $a_k$ are the coefficients (Burg 1967). $P_{M+1}$ is connected to the $(M+1)$ filter coefficients by the following linear system

$$\begin{pmatrix} \phi_0 & \phi_1 & \ldots & \phi_M \\ \phi_1 & \phi_0 & \phi_1 & \ldots & \phi_{M-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \phi_M & \phi_{M-1} & \ldots & \phi_0 \end{pmatrix} \begin{pmatrix} 1 \\ -a_1 \\ \vdots \\ -a_M \end{pmatrix} = \begin{pmatrix} P_{M+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$
\( \phi_k \) are the autocovariance estimates with lag \( k \) (Peacok & Treitel 1969). Many authors have shown the connection between the Maximum Entropy Spectral Analysis (MESA) and the Autoregressive Representation (AR) of a stationary stochastic process (see, for example, Van den Bos 1971; Ulych & Bishop 1975; Ulych & Clayton 1976). Indeed equation (1) gives also the spectrum of an AR stochastic process of order \( M \). Ulych & Bishop (1975) have shown two approaches can be used to compute \( a_k \). The first one is useful when \( \phi_k \) are known and computes \( a_k \) by the linear system (Yule–Walker equations)

\[
\begin{pmatrix}
\phi_0 & \phi_1 & \ldots & \phi_{M-1} \\
\phi_1 & \phi_0 & \phi_1 & \ldots & \phi_{M-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\phi_{M-1} & \phi_{M-2} & \ldots & \phi_0
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_M
\end{pmatrix}
= 
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_M
\end{pmatrix}
\]

In this case the autocovariances are estimated by the data set \( x_1, x_2, \ldots, x_N \)

\[
\phi_k = \frac{1}{N} \sum_{j=1}^{N-k} (x_{j+k} - \langle x \rangle) (x_j - \langle x \rangle)
\]

the brackets indicate the mean value, and \( a_k \) can be computed by the Levinson (1947) algorithm using the properties of the Toeplitz matrix of autocovariances. The preceding estimate of autocovariances (3) implies, however, that \( x_j = 0 \) for \( j > N \), and this does not respect the maximum entropy principle (Burg 1967).

In the second one, prior estimation of the autocovariances is not necessary: Burg (1968) has suggested an iterative method to compute \( a_k \) directly from the input data set. In this case, no hypothesis is advanced concerning autocovariances for time lags exceeding the total duration of the data set, and the principle of maximum entropy is respected. The values of \( \phi_k \) are calculated iteratively once the \( a_k \) and \( \phi_0 \) are known

\[
\phi_k = \sum_{j=1}^{k} a_k \phi_{j-k}.
\]

This last approach is generally used: Andersen (1974) has given a fast algorithm for computing \( a_k \). Other authors (see for example Ulych & Clayton 1976) have shown that least-square techniques minimize the prediction-error power in comparison with the Burg estimate, and are more stable with respect to the initial phase of the signal (see also Chen & Stegen 1974). Using this method, Barrodale & Erickson (1980a, b) have developed an algorithm for calculating maximum entropy spectra by a least-squares linear prediction directly without applying a Toeplitz structure.

In all these different approaches, a very delicate problem, for which so far there is no exact rule on a physical basis, is the choice of the length of filter (\( M \) in equation 1). On a mathematical basis, the most frequently applied criterion consists in estimating the order of the autoregressive series that models the processes, but this has no clear physical meaning. Haykin & Kesler (1979) give a review of some of the more common methods for choosing \( M \) using this criterion. These methods consist in taking \( M \) as the number for which one of the following quantities is a minimum:
Correlation time in MESA

(a) Final prediction error (Akaike 1969, 1970)

\[ FPE(M) = \frac{N - (M + 1)}{N + (M + 1)} P_M. \]

(b) Information theoretic criterion (Akaike 1974, 1976)

\[ AIC(M) = \ln P_M + \frac{2M}{N}. \]

(c) Autoregressive transfer function criterion (Parzen 1976)

\[ CAT(M) = \frac{1}{N} \sum_{j=1}^{M} \frac{N - j}{NP_j} - \frac{N - M}{NP_M}. \]

These criteria give good results when applied to numerically simulated autoregressive processes.

Conversely, when they are used to study real geophysical data sets, they generally give values of \( M \) which are too small (Ulrych & Bishop 1975 and Section 3). Also Landers & Lacoss (1977) find too low values of \( M \) in harmonic processes perturbed by strong noise. Ulrych & Bishop (1975) have shown heuristically that the spectra calculated with \( M = N/2 \) are good for short records. Berryman (1978) has shown that the choice \( M = 2N/\ln 2N \) is better than \( M = N/2 \) because \( M = N/2 \) may lead to numerous instabilities in the spectrum. Ulrych & Clayton (1976) and Ulrych & Ooe (1979) give as an empirical rule for \( M \)

\[ \frac{N}{3} \leq M \leq \frac{N}{2}. \]

These kinds of \textit{a priori} choice give \( M \) as a function of \( N \). Such a choice has a practical justification: the spectrum shows a correct compromise between over-smoothing behaviour (low value of \( M \)) and behaviour with instabilities (high value of \( M \)). As far as the choice of optimum length is concerned, the criteria (a), (b) and (c) can be said to be efficient for processes which have a strong autoregressive character but do not provide good results very often for real geophysical data: the \textit{a priori} choice of \( M \) as a function of \( N(N/3 < M < N/2 \) or \( M = 2N/\ln 2N) \) gives satisfactory results for physical processes, but has the severe limitation of not taking into account the stochastic properties peculiar to the time data under analysis. Treitel, Gutowski & Robinson (1976) suggest to compute \( M \) by studying the behaviour of \( |a_M| \) versus \( M \). They show that \( |a_M| \) often falls sharply at a certain value of \( M \); they also observe that the value of \( M \) for which \( |a_M| \) has the steep roll-off is practically the same as that which presents a minimum of the square deviation between the real spectrum of the process and the theoretical spectrum as a function of \( M \). This last criterion is good for AR stochastic processes, but when it is used to analyse real geophysical data, the behaviour of \( |a_M| \) is not stable, and the criterion cannot be applied. Besides, it is obviously not possible to obtain the real process spectrum in the case of geophysical phenomena.

As far as the conceptual starting point is concerned, our approach is similar to Treitel’s: the length of the filter must be such that it contains all, and only, the physical information (Rovelli 1982). The aim of this paper is to suggest a new criterion for computing \( M \) on the basis of the intrinsic ‘memory time’ of the data set, which is provided by the autocovariance sequence. This method furnishes better performances when applied to real geophysical data.
The characteristic correlation time

The fundamental point of our criterion is that the MESA filter length must be conceptually linked with an easily calculable quantity, the characteristic correlation time (CCT) of time series. CCT is defined as follows

\[
CCT = \Delta t \sum_k \frac{|\phi_k|}{\phi_0},
\]

the sum is extended as far as the maximum time lag for which the autocovariances are computed. The expression (5) is derived from the definition of the correlation time-scale \(\lambda\) of a continuous stochastic process \(x(t)\) (Panchev 1971)

\[
\lambda = \frac{1}{\phi_0} \int_0^{T_0} |\phi(t')| \, dt'
\]

where

\[
\phi(\tau) = \frac{1}{T_0} \int_0^{T_0} x(t' + \tau) x(t') \, dt' - \langle x \rangle^2.
\]

\(T_0\) is the time length of the process. The time-scale of correlation is the characteristic memory time of the process, i.e., \(x(t_0)\) and \(x(t_0 + t)\) are practically uncorrelated for \(t \geq \lambda\). The importance of \(\lambda\) can be seen also from the following relation (Panchev 1971)

\[
\frac{\langle (\bar{x}^T - \langle x \rangle)^2 \rangle}{\langle (x - \langle x \rangle)^2 \rangle} \leq \frac{\lambda}{T}
\]

where \(\bar{x}^T\) is the mean value as time average

\[
\bar{x}^T = \frac{1}{T} \int_0^T x(t') \, dt'
\]

and \(\langle x \rangle\) is the 'true' mean value. From equation (6) \(\lambda\) may be said to give the convergence time of \(\bar{x}^T\) to \(\langle x \rangle\).

Our idea of choosing \(M\) is based on the fact that the MESA filter length must be proportional to CCT. This is justified by the fact that \(x(t_0)\) and \(x(t_0 + t)\) are correlated when \(t \leq \text{CCT}\); therefore a choice of \(M < \text{CCT}/\Delta t\) will give a very poor spectrum because the filter has not all the information pertaining to the process. On the other hand if one chooses \(M\) too large, non-physical effects are included as the filter contains more than just physical information. The result is a large number of peaks in the spectrum due to poles of equation (1), i.e., when \(M \gg \text{CCT}/\Delta t\) the long suite of the filter coefficients does not add further information but only uncorrelated errors. Following the previous arguments, the quantity \(\text{CCT}/\Delta t\) may be said to be a useful estimator for \(M\) determination. In many processes of geophysical interest \(\phi_1\) has the form

\[
\phi_1 = \phi_0 \exp(-t/\tau) \cos(2\pi f_0 t).
\]

This kind of process is defined as red noise (see, for example, Bath 1974). In this case, it is possible to have analytically an estimation of CCT

\[
\text{CCT} = \int_0^{T_0} \exp(-t/\tau) |\cos(2\pi f_0 t)| \, dt = \int_0^{\infty} \exp(-t/\tau) |\cos(2\pi f_0 t)| \, dt
\]

\[
= \frac{1}{\tau^{-1} + 4\pi^2 f_0^2} \left[ \tau^{-1} + \frac{4\pi f_0 \exp[-1/(4f_0 \tau)]}{1 - \exp[-1/(2f_0 \tau)]} \right]
\]

for sufficiently large \(T_0\).
Figure 1. CCT curves for red noise obtained with $\tau$ equal to different percentages of total length $T_0$ and for frequencies $0 < f_0 < 1/2\Delta t$. A numerically simulated autocovariance sequence has been used $\phi_t = \phi_0 \exp(-t/\tau) \cos(2\pi f_0 t)$, with $t = 0, 1, 2 \ldots T_0$ and $f_0 = 2\pi/k$, putting $T_0 = 180$ and $K = 180, 179, \ldots, 3, 2$ ($\Delta t$ is supposed equal to 1). As can be seen, CCT is virtually $2\tau/\pi$ in realistic cases over the range $2/T_0 \leq f_0 < 1/2\Delta t$.

When $f_0 \tau \gg 1$, CCT is

$$\text{CCT} \approx \frac{2\tau}{\pi}.$$ 

We can see in Fig. 1 CCT behaviour as a function of $f_0$ for $\tau$ equal to different percentages of $T_0$. Actually, in practice (i.e. $\tau$ up to a few per cent of $T_0$ and $f_0$ 2 or 3 times greater than $1/T_0$) expression (8) takes on a value of roughly $2\tau/\pi$. It follows by definition of $\tau$ that a time of about $\tau$ is required for practically all the information to be obtained without noise terms: for red noise we give

$$|M\Delta t = \tau|.$$ 

Therefore

$$M = \frac{\pi}{2\Delta t} \text{CCT} = \frac{\pi}{2} \frac{\Sigma_t |\phi_t|}{\phi_0}.$$ 

$M$ is in this formula of sufficient length to contain all the predictable information present in the data. An expression has thus been obtained for writing $M$ for red noise by means of an easily computable quantity

$$\frac{\Sigma_t |\phi_t|}{\phi_0}.$$ 

It also takes the degree of intrinsic data set correlation into account. It follows by definition of CCT that this formula of $M$ is reasonable also for processes with autocovariances different from (7), this choice being realistic also for series with strong Markovian behaviour in the broader sense.
To test the validity of this criterion some analyses have been carried out (Section 3) on real geophysical data as well as numerical simulations of stochastic processes (white noise) and harmonic processes with additive noise.

3 Applications to real geophysical data and numerical simulations

In this section we shall discuss some geophysical processes and numerically simulated time series which have been analysed in order to test the CCT criterion for the optimum MESA filter length

\[ M_{\text{op}} = \frac{\pi}{2} \sum_{r=1}^{N} |\phi_r| \]

Since, for completely uncorrelated series, the CCT must, by definition, be zero (see case (f) below), the \( M_{\text{op}} \) expression is rewritten as follows:

\[ M_{\text{op}} = \frac{\pi}{2} \sum_{r=1}^{N} |\phi_r| \]

\[ (9) \]

Figure 2. Spectra calculated in the frequency band from 0 to 0.5\( f_N \), with \( f_N = 1/2\Delta t \), for the real geophysical processes illustrated in cases (a), (b), (c), (d) and (e), corresponding to MESA filters with \( M \) length equal to 10, 20, ..., 80 per cent of the total data \( N \). The spectrum calculated for the value obtained using the CCT criterion is indicated by the arrow.
Correlation time in MESA

Figure 3. Burg’s autocovariance estimates for the processes in cases (a), (b), (c) and (d) used to calculate $M_{DP}$ by CCT criterion.

Thus guaranteeing the zero CCT condition for uncorrelated sequences. For any series of geophysical data, each one of a different kind and a different degree of autocorrelation, we give the behaviour of spectral density for various $M$ values: we show (see Figs 2 and 8) spectral densities corresponding to filter length equal to 10, 20, ..., 80 per cent of total number of data $N$. For any process, the arrows point to the spectral density computed with the filter length whose value is nearest to one calculated by equation (9). For any series of data we show also the value of $M$ given by the other criteria (FPE, AIC, CAT), see Table 1. Burg’s autocovariance estimates $\phi_t$, computed by (4), are shown in Figs 3 and 7.

The argument that the statistical reliability of autocovariance estimates decreases with increasing time lags deserves special mention. With a long series of data available it is, however, possible to compute the process autocovariances with a high degree of statistical reliability. Then, if the known series is truncated after a short interval so that only a small part of the process is considered and the whole of the remaining part is neglected, and if Burg’s estimates are computed for the whole of this short interval, it becomes possible to

Table 1. Sample parameters of data utilized in this work.

<table>
<thead>
<tr>
<th></th>
<th>Length of series $N$</th>
<th>Sampling rate $\Delta t$</th>
<th>FPE criterion</th>
<th>AIC criterion</th>
<th>CAT criterion</th>
<th>$\frac{\sum_t \phi_t}{\phi_0}$</th>
<th>CCT criterion</th>
<th>Percentage of total length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sunspot numbers</td>
<td>204</td>
<td>1 yr</td>
<td>8</td>
<td>3</td>
<td>9</td>
<td>41</td>
<td>65</td>
<td>32</td>
</tr>
<tr>
<td>Internal wave</td>
<td>60</td>
<td>2 hr</td>
<td>21</td>
<td>3</td>
<td>21</td>
<td>13</td>
<td>21</td>
<td>37</td>
</tr>
<tr>
<td>Long-period noise</td>
<td>128</td>
<td>2 s</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>16</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>Nuclear explosion</td>
<td>180</td>
<td>0.5 s</td>
<td>14</td>
<td>3</td>
<td>14</td>
<td>22</td>
<td>34</td>
<td>19</td>
</tr>
<tr>
<td>Tidal current velocity</td>
<td>120</td>
<td>1 hr</td>
<td>16</td>
<td>4</td>
<td>16</td>
<td>35</td>
<td>54</td>
<td>46</td>
</tr>
<tr>
<td>White noise</td>
<td>150</td>
<td>1 time unit</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>Harmonic signal with</td>
<td>150</td>
<td>1 time unit</td>
<td>15</td>
<td>6</td>
<td>14</td>
<td>93</td>
<td>145</td>
<td>97</td>
</tr>
<tr>
<td>additive noise</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
make a comparison between the practically 'true' autocovariances of the process and the estimates we can perform using Burg's algorithm over the short partial interval of the process under investigation.

By means of short-period seismic noise and geomagnetic declination digital recordings Rovelli (1982) has shown that the 'true' autocovariances \( \hat{\phi}_t \) obtained by the following formula

\[
\hat{\phi}_t = \frac{1}{N} \sum_{j=1}^{N-t} x_j x_{j+t}
\]

with \( t = 0, 1, 2 \ldots t_{\text{max}} \) computed up to a time lag \( t_{\text{max}} \) corresponding a certain percentage of \( N \) (12 per cent for the seismic noise, 25 per cent for the geomagnetic declination) do not differ considerably from Burg's estimates \( \phi_t^B \)

\[
\phi_t^B = \sum_{j=1}^{t_{\text{max}}} a_j \hat{\phi}_{t-j}
\]

for \( t = 1, 2 \ldots t_{\text{max}} \), \( \hat{\phi}_t^B = \hat{\phi}_0 = \phi_0 \) computed on the reduced data set of \( t_{\text{max}} \) samples only. In Rovelli (1982) one can see that in the second half of the time lags interval \((0, t_{\text{max}})\), Burg's estimates begin to differ quite appreciably from the 'true' autocovariances. However, what does not appear to be significant is the error introduced in summation (9). Really, the error in computing the filter length by means of Burg's estimates is very low, even taking into consideration time lags for which the statistical reliability is not good. This is due to the fact that the essential contribution to the summation is made by the terms corresponding to low time lag values which do not differ from the 'true' estimates. It would thus seems legitimate to use the summation (9) to calculate the correlation time even when the terms of the summation are not statistically reliable for high time lag values

\[
M_B = \frac{\pi}{2} \frac{\sum_{t=1}^{t_{\text{max}}} |\phi_t^B|}{\phi_0} \approx \hat{M} = \frac{\pi}{2} \frac{\sum_{t=1}^{t_{\text{max}}} |\hat{\phi}_t|}{\phi_0}.
\]

As an example see Fig. 4 \((M_B = 29, \hat{M} = 27)\).

In the following we shall analyse some time series. In each of these cases, more accurate results may be obtained by carrying out preliminary elaborations on the data sets. However, our goal is to discuss CCT criterion, and for more specific results the reader is referred to the references.

### 3.1 Sunspot Numbers

It is a very well studied historical sequence. Yule (1927) was the first to analyse the periodicities of sunspot numbers of Wolfer's series using an autoregressive model. Bloomfield (1976) has applied many spectral techniques to the detection of the frequencies of Wolf's sunspot numbers. Currie (1973) gives a detailed description of MESA spectrum in the band between 2 and 70 yr. Fougere, Zawaliok & Radoski (1976) have proved the stability of the spectrum changing initial phase of the series by running over the whole sequence a window with a length of 195 yr. Fig. 2(a) shows the results of our analysis using Wolf's data from 1749 to 1953. The spectral density is quite sharp when \( M/N = 30 \) per cent (Fig. 2a, picture indicated by arrow) and the periods of 74.2, 11.0, 9.8 and 8.3 yr are particularly peaked. It is interesting to note that in this case CCT criterion gives \( M = 0.32 N \) while too low values of \( M \) are given by the other criteria (Table 1).
3.2 INTERNAL WAVES IN THE LIGURIAN SEA

The time fluctuations of seawater temperature at the thermocline depth (~70 m) observed in a fixed station in the Ligurian Sea (Colacino et al. 1979) are studied by MESA. The spectral densities, with $M$ computed by (9) (picture with arrow in Fig. 2b), show two very sharp peaks corresponding to periodicities of 140 and 30 hr. It is also possible to see the effects of tides (8 and 12 hr). The least squares method, applied after low-pass filtering (see Fig. 5), gives the following values for the amplitudes $A$ of the two principal modes

\[ A(140) = 0.48^\circ C \]
\[ A(30) = 0.57^\circ C. \]

The result is qualitatively in good agreement with the spectrum indicated by arrow in Fig. 2(b). Fig. 5 shows the fitting with these two periods is really a good approximation of experimental data.

3.3 LONG-PERIOD SEISMIC NOISE

A long-period noise record of the Monteporzio seismic station (Rome) is spectrally analysed. In Fig. 2(c) the periodicities of the long-period noise (13.4 and 9.8 s) are clear enough in the spectrum computed with 20 per cent of $N$ as the CCT criterion requires.

3.4 NUCLEAR EXPLOSION SEISMIC RECORDING

The signal recorded by the seismic station of Montasola (Italy) on 1980 September 18 is processed by the same methods as those used in the previous cases. Also in this case the estimate of $M$ obtained by the CCT criterion is better than those obtained using other criteria (Table 1).

3.5 TIDAL WATER CURRENT VELOCITY IN THE MESSINA STRAIT

Vercelli's classic measurements (Vercelli 1925; Vercelli & Picotti 1925) of the water velocity in the Messina Strait still give useful information about the strait dynamics and tidal currents.
Figure 5. The full-line curve shows the behaviour of the seawater temperature at the thermocline depth (~70 m) after processing with a low-pass filter which, in the presence of the stratification shown at the top of the figure, corresponds to an internal wave with an amplitude of about 10 m. The dotted line curve was obtained using the least-squares method and superimposing two oscillation modes of 140 and 30 hr. The agreement between the two curves is such that the spectrum indicated by the arrow can be considered to be the best estimate of the (b) spectra in Fig. 2.

Figure 6. Behaviour of the MESA spectrum of the stream velocity of the water in the Messina Strait calculated for \( M = 0.4 N \) (a) and \( M = 0.5 N \) (b). The line of the main tidal is split, as \( M \) approaches the value given by the CCT criterion, into the two lines \( M_1 \) (theoretically 12.42 hr) and \( S_1 \) (theoretically 12.00 hr).
Two problems in WKBJ theory

Irn (p)

Figure 7. (a) The cut p-plane. \( z_p \) is multivalued even along \( \Gamma \); (b) the associated contour \( z_p(\Gamma) \). \( z_p \) jumps from \( z^* \) to \( z_2 \) but the contour of integration for \( \tau \) still passes through \( z_3, z_4 \), and \( z_5 \).

the unique point between \( z_5 \) and \( z_6 \) (unique because we specify that the \( z_5 - z_6 \) velocity regime be used) at which \( u(z_5, \omega) = u_2(\omega) \). If we are using the linear interpolation scheme (7) then:

\[
z_\phi = z_5 + \frac{v_2 - v_5}{v_6 - v_5} (z_6 - z_5). \quad (25)
\]

It can be seen that WKBJ methods with only one turning point assume that no energy tunnels through the high-velocity region between \( z_1 \) and \( z_4 \). This tunneled energy is insignificant for wavelengths much less than \( z_4 - z_1 \) but is significant otherwise. When the two turning points are close together, i.e. when \( p \) is close to \( 1/v_2 \) in Fig. 7(a), then this energy can be estimated accurately only by approximations in terms of parabolic cylindrical functions (Kennett & Illingworth 1981). The alternative procedure, outlined in the next section, is less accurate but perhaps simpler.

5 Tunnelling

We have seen above how the interpolation of a sampled velocity function leads to the definition of velocity in a cut \( z \)-plane. In tunnelling problems this is an advantage since it simplifies the analytic continuation of velocities and leads naturally to a solution in terms of Airy functions (or equivalently, Hankel functions of order 1/3). To show how this solution is obtained we use as an example the velocity function of Fig. 6(a) and the linear interpolation scheme (7). This velocity function is reproduced in Fig. 8 in which, so that the results given here may be applied elsewhere, we have distinguished the depth points \( z_2 \) and \( z_4 \) by renaming them \( z^- \) and \( z^+ \), respectively. These two points divide the Re\( (z) \)-axis into three ranges in which \( u(z) \) is monotonic. By linear extrapolation each monotonic velocity function can be extended to all depths. With each velocity function there are associated two independent solutions of the wave equation, of the form (5). Any solution to the wave equation for \( v = u(z) \) will be a linear combination of the solutions to one of the monotonic velocity functions for the depth range in which that monotonic velocity function coincides with \( u(z) \). The most convenient wavefunctions in the middle region, where \( u(z) \) is decreasing, are obtained by reversing the order of integration in the definition of \( \tau \), equation (4). This is
4 Conclusions

In this approach to the choice of the MESA filter we have tried to obviate the shortcoming that the standard criteria are unfit when applied to real geophysical data. In these cases, we have shown that physical basis considerations that utilize the information of the intrinsic data set furnish preferable results if compared to the ones obtainable using the criteria related to the order of the AR process that models the data. It must be underlined that the CCT criterion cannot be utilized for obtaining an estimate of the AR process order, and it is not applicable to particular mathematical series having autocovariances with unusual behaviours (e.g. sparse processes, Cleveland, private communication). But this kind of sequences is not realistic in physical cases.

From the previous results on the processed series it is possible to derive the following remarks:

1) The application of equation (9) gives the best results for all the geophysical data analysed;
Figure 9. Amplitude of the last of the MESA filter coefficients $|a_M|$ with varying filter length $M$. In none of the processes illustrated is there any perceptible fall-off from a certain value of $M$ on, as does happen in processes constructed on the basis of AR models (Treitel et al. 1976).

(2) the CCT criterion furnishes too large values of $M(M \sim N)$ only for harmonic signals, when the autocovariance envelope is practically not decreasing for increasing time lags (theoretical limit of 7 when $\tau \to \infty$): for this kind of non-physical signals the CCT criterion application is not advisable, even if the spectrum obtained is still realistic;

(3) Treitel's criterion, when used to analyse geophysical data, does not give a profile of $|a_M|$ versus $M$ with a sharp roll-off (Fig. 9);

(4) the spectra computed with $M$ estimated by means of the other criteria (FPE, AIC, CAT) appear often very smooth for geophysical data, and consequently are quite poor in information.

The proposed method is not data-adaptive; it is not an a priori method because it gives the value of the MESA filter length directly by (9) and so $M_{op}$ is not explicitly a function of $N$; CCT criterion takes into account the statistical properties of the time series under analysis and uses all the information contained in the data.

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References


