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Luni-Solar Perturbations of the Orbit of an Earth Satellite

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Summary

The effects of the gravitational attractions of the Sun and Moon on the orbital elements of an Earth satellite are investigated using Lagrange's planetary equations. Expressions are obtained for the change in the elements during one revolution of the satellite and for the rates of change of these elements. Corresponding expressions are obtained for the effects of solar radiation pressure, including the effect of the Earth's shadow.

1. Introduction

The effects of the gravitational attractions of the Sun and Moon on the orbits of artificial Earth satellites are now important for two reasons. Firstly, they must be included in the analysis when the harmonics in the Earth's gravitational potential are evaluated from satellite observations (King-Hele 1961), and, secondly, they are important for satellites which travel out to several Earth radii, e.g. Explorer 6 or various proposed communication satellites. The force due to solar radiation is greater than air drag for all satellites above about 1 000 km; the effects of solar radiation pressure, however, are mainly important for balloon-type satellites, e.g. Echo.

The effect of a disturbing body on the Moon's orbit has been studied for many years, but the resulting lunar theory is not applicable to artificial satellites. Luni-solar perturbations of the orbits of artificial satellites have been studied recently in several papers: all these papers are, however, subject to certain limitations.

The first paper, by Spitzer (1950), uses only the first terms of the Hill-Brown lunar theory, so that the results are limited by the assumptions of small eccentricity and small inclination of the orbit of the disturbed body to the orbit of the disturbing body, which were used in the development of that theory.

In the second paper, Kozai (1959) writes down Lagrange's planetary equations and the disturbing function due to the Sun or Moon, including both secular and long period terms, but he only gives explicit expressions for the secular terms.

In the third paper, Blitzer (1959) ignores the specialized techniques of celestial mechanics and obtains estimates of the perturbations by the methods of classical mechanics. Again, only secular terms are included.

In the fourth paper, Moe (1960) also uses Lagrange's planetary equations, but chooses a co-ordinate system in which one axis is along the orbital angular momentum vector of the disturbing body. For a near Earth satellite, however, the most useful set of axes has one axis coincident with the Earth's axis.

Geyling (1960) writes down the equations of motion in terms of the displacement components relative to the unperturbed elliptic orbit, but only solves them for a circular orbit.

Papers by Musen (1960a) and Upton, Bailie & Musen (1959) also discuss luni-solar perturbations although they do not give general results, but only the effects on particular satellites. They do, however, point out the possibility of resonance effects.

In this paper, perturbations due solely to a third body are determined from Lagrange's planetary equations by integrating over one revolution of the satellite. The rates of change of the orbital elements averaged over one revolution are then written down. All first-order terms, both secular and long-period (greater than the period of revolution of the satellite), are retained in the analysis. There is no limitation on the eccentricity, but, owing to the neglect of high-order terms in r/r_d, where r and r_d are the radial distances from the Earth of the satellite and disturbing body respectively, the theory is limited to satellites whose semi-major axis does not exceed one tenth of the Moon's distance from the Earth.

Corresponding expressions are obtained for the changes in the orbital elements due to solar radiation pressure. When the effect of the Earth's shadow is neglected, the results reduce to those given by Musen (1960b).

2. Lagrange's planetary equations

The notation is illustrated in Figure 1, which shows the projection of the satellite orbit on the unit sphere. The components of the perturbing force per unit mass are denoted by S, T and W, where S is along the radius vector from the Earth's centre, T is perpendicular to S and in the osculating plane of the orbit, and W is normal to the osculating plane. The osculating plane is defined as the plane containing the satellite's velocity vector and passing through the Earth's centre. Its position is defined by the right ascension of the ascending node Ω and by the inclination i, defined as the angle ($0 \leqslant i \leqslant \pi$) between the eastward direction at the equator and the osculating plane at the ascending node N. The position of the osculating ellipse in the orbital plane is defined by the argument of perigee, ω. The size of the orbit is specified by the semi-major axis, a, and the shape by the eccentricity, e. The position of the satellite in the ellipse is defined by the angle u, as measured from the ascending node, and by its radial distance r from the Earth's centre.

Lagrange's planetary equations express the rate of change of the osculating elements in terms of the components of the perturbing force. These equations are (Tisserand 1889, p. 433):

$$\dot{a} = \frac{2}{n(1-e^2)^{\frac{1}{2}}}\left[S\, e \sin\theta + \frac{a}{r}(1-e^2)T\right] \tag{1}$$

$$\dot{e} = \frac{(1-e^2)^{\frac{1}{2}}}{na}[S\sin\theta + T(\cos\theta+\cos E)] \tag{2}$$

$$\frac{di}{dt} = \frac{Wr\cos u}{na^2(1-e^2)^{\frac{1}{2}}} \tag{3}$$

$$\dot{\Omega} = \frac{Wr\sin u}{na^2(1-e^2)^{\frac{1}{2}}\sin i} \tag{4}$$

$$\dot{\omega} = \frac{(1-e^2)^{\frac{1}{2}}}{nae}\left[-S\cos\theta + \left\{1+\frac{r}{a(1-e^2)}\right\}T\sin\theta - \frac{er}{a(1-e^2)}W\cot i\sin u\right] \tag{5}$$

$$\frac{dt}{du} = \frac{\gamma r^2}{na^2(1-e^2)^{\frac{1}{2}}}, \tag{6}$$

where (Tisserand 1889, p. 462)

$$\frac{1}{\gamma} = 1 - \frac{Wr^3\cot i\sin u}{n^2a^4(1-e^2)}, \tag{7}$$

n is the mean angular motion of the satellite, θ is the true anomaly and E is the eccentric anomaly.

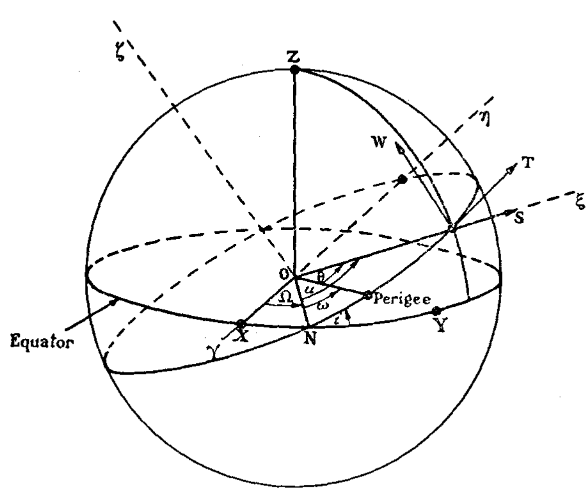

FIG. 1.—Projection of satellite orbit on unit sphere, showing notation.

Two simplifications in these equations are possible, since the main changes in Ω and ω are those due to the Earth's gravitational field (King-Hele 1958). Firstly, since $d\Omega/du$ is of the order of $10^{-3}\cos i$, equation (7) can be written as

$$\gamma = 1 + \frac{d\Omega}{du}\cos i = 1 + O(10^{-3}). \tag{8}$$

Secondly, since $u = \theta+\omega$, it is possible to write

$$\frac{du}{d\theta} = 1+\frac{d\omega}{d\theta} = 1+O\left(\frac{1}{300}\right) \tag{9}$$

as $d\omega/d\theta$ is of the order of $10^{-3}(5\cos^2 i-1)$. These simplifications are still valid when luni-solar perturbations are included, as these are $O(10^{-5})$ or less in the range of validity of the present theory.

Using equations (6), (8) and (9), the derivatives in (1) to (5) may be replaced by derivatives with respect to θ:

$$\frac{da}{d\theta} = \frac{2r^2}{n^2a^2(1-e^2)}\left[Se\sin\theta+\frac{a}{r}(1-e^2)T\right] \tag{10}$$

$$\frac{de}{d\theta} = \frac{r^2}{n^2a^3}[S\sin\theta+T(\cos\theta+\cos E)] \tag{11}$$

$$\frac{di}{d\theta} = \frac{Wr^3\cos u}{n^2a^4(1-e^2)} \tag{12}$$

$$\frac{d\Omega}{d\theta} = \frac{Wr^3\sin u}{n^2a^4(1-e^2)\sin i} \tag{13}$$

$$\frac{d\omega}{d\theta} = \frac{r^2}{n^2a^3e}\left[-S\cos\theta+\left(1+\frac{r}{a(1-e^2)}\right)T\sin\theta - \frac{er}{a(1-e^2)}W\cot i\sin u\right]. \tag{14}$$

The changes in the orbital elements are obtained by integrating equations (10) to (14) over one revolution of the satellite, assuming that the orbital elements remain constant during the time of integration and that the radial distance of the satellite is the same as for an unperturbed orbit. The components of the disturbing force are derived in Section 3.1 and the integration is performed in Section 3.2. The effect of solar radiation pressure is dealt with in Section 4.

The period of revolution of the satellite, which is the same as for an unperturbed orbit, is

$$\frac{2\pi a^{\frac{3}{2}}}{(GM_e)^{\frac{1}{2}}} \text{ or } \frac{2\pi}{n}, \tag{15}$$

where G is the constant of gravitation and M_e is the mass of the Earth.

3. Perturbations due to the gravitational attraction of a third body

3.1. *The disturbing force due to a third body*

Let O be the origin of a co-ordinate system $OXYZ$ such that OX is the line from which Ω is measured and OZ is directed northwards along the Earth's axis, as shown in Figure 1; let the co-ordinates of the satellite be (x, y, z) and those of the disturbing body be (x_d, y_d, z_d). Then the disturbing function R is given by (Smart, 1953, p. 9)

$$R = GM_d\left(\frac{1}{\Delta}-\frac{xx_d+yy_d+zz_d}{r_d^3}\right), \tag{16}$$

where M_d is the mass of the disturbing body, which is at a distance r_d from the Earth's centre. Δ denotes the distance between the satellite and the disturbing body and is given by

$$\Delta^2 = (x-x_d)^2+(y-y_d)^2+(z-z_d)^2. \tag{17}$$

Differentiation of the disturbing function gives the components of the disturbing force along the axes OX, OY and OZ. These are:

$$\left.\begin{aligned}\frac{\partial R}{\partial x} &= -GM_d\left(\frac{x-x_d}{\Delta^3}+\frac{x_d}{r_d^3}\right)\\ \frac{\partial R}{\partial y} &= -GM_d\left(\frac{y-y_d}{\Delta^3}+\frac{y_d}{r_d^3}\right)\\ \frac{\partial R}{\partial z} &= -GM_d\left(\frac{z-z_d}{\Delta^3}+\frac{z_d}{r_d^3}\right).\end{aligned}\right\} \tag{18}$$

Let the lines $O\xi$, $O\eta$ and $O\zeta$ of Figure 1, which are parallel to S, T and W, have direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) respectively with reference to the axes OX, OY and OZ. In terms of the satellite's co-ordinates these are:

$$\left.\begin{aligned}l_1 &= \cos\Omega\cos u-\sin\Omega\sin u\cos i\\ m_1 &= \sin\Omega\cos u+\cos\Omega\sin u\cos i\\ n_1 &= \sin u\sin i\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}l_2 &= -\cos\Omega\sin u-\sin\Omega\cos u\cos i\\ m_2 &= -\sin\Omega\sin u+\cos\Omega\cos u\cos i\\ n_2 &= \cos u\sin i\end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned}l_3 &= \sin\Omega\sin i\\ m_3 &= -\cos\Omega\sin i\\ n_3 &= \cos i.\end{aligned}\right\} \tag{21}$$

If the orbital elements of the disturbing body, denoted by the suffix d, are defined in the same way as those of the satellite, its co-ordinates are given by

$$\left.\begin{aligned}x_d &= r_d(\cos\Omega_d\cos u_d-\sin\Omega_d\sin u_d\cos i_d)\\ y_d &= r_d(\sin\Omega_d\cos u_d+\cos\Omega_d\sin u_d\cos i_d)\\ z_d &= r_d\sin u_d\sin i_d.\end{aligned}\right\} \tag{22}$$

Denoting the angle between the radius vector to the satellite and the disturbing body by ϕ, we can write

$$\Delta^2 = r^2+r_d^2-2r\,r_d\cos\phi, \tag{23}$$

so that

$$\frac{1}{\Delta^3} = \frac{1}{r_d^3}\left[1+3\frac{r}{r_d}\cos\phi+O\left(\frac{r^2}{r_d^2}\right)\right]. \tag{24}$$

Using (19) and (22), we obtain

$$\cos\phi = \frac{xx_d+yy_d+zz_d}{rr_d} = A\cos u+B\sin u, \tag{25}$$

where

and

$$\left.\begin{aligned} A &= \cos(\Omega-\Omega_d)\cos u_d+\cos i_d \sin u_d \sin(\Omega-\Omega_d) \\ B &= \cos i[-\sin(\Omega-\Omega_d)\cos u_d+\cos i_d \sin u_d \cos(\Omega-\Omega_d)] \\ &\qquad +\sin i \sin i_d \sin u_d. \end{aligned}\right\} \tag{26}$$

Substitution of equation (24) into (18) gives

$$\left.\begin{aligned} \frac{\partial R}{\partial x} &= -\frac{GM_d r}{r_d^3}\left[l_1+3\left(l_1\frac{r}{r_d}-l_d\right)\cos\phi+\frac{3}{2}l_d\frac{r}{r_d}(1-5\cos^2\phi)\right] \\ \frac{\partial R}{\partial y} &= -\frac{GM_d r}{r_d^3}\left[m_1+3\left(m_1\frac{r}{r_d}-m_d\right)\cos\phi+\frac{3}{2}m_d\frac{r}{r_d}(1-5\cos^2\phi)\right] \\ \frac{\partial R}{\partial z} &= -\frac{GM_d r}{r_d^3}\left[n_1+3\left(n_1\frac{r}{r_d}-n_d\right)\cos\phi+\frac{3}{2}n_d\frac{r}{r_d}(1-5\cos^2\phi)\right]. \end{aligned}\right\} \tag{27}$$

The components of the perturbing force, as defined in Section 2, are given in terms of the disturbing function by

$$\left.\begin{aligned} S &= l_1\frac{\partial R}{\partial x}+m_1\frac{\partial R}{\partial y}+n_1\frac{\partial R}{\partial z} \\ T &= l_2\frac{\partial R}{\partial x}+m_2\frac{\partial R}{\partial y}+n_2\frac{\partial R}{\partial z} \\ W &= l_3\frac{\partial R}{\partial x}+m_3\frac{\partial R}{\partial y}+n_3\frac{\partial R}{\partial z}. \end{aligned}\right\} \tag{28}$$

Substituting (27) into (28) gives

$$\left.\begin{aligned} S &= -Kr\left[1+3\left(\frac{r}{r_d}-\cos\phi\right)\cos\phi+\frac{3}{2}\frac{r}{r_d}(1-5\cos^2\phi)\cos\phi\right] \\ T &= 3Kr(l_2l_d+m_2m_d+n_2n_d)\left[\cos\phi-\frac{1}{2}\frac{r}{r_d}(1-5\cos^2\phi)\right] \\ W &= 3Kr(l_3l_d+m_3m_d+n_3n_d)\cos\phi\left[-\frac{1}{2}\frac{r}{r_d}(1-5\cos^2\phi)\right] \end{aligned}\right\} \tag{29}$$

where $K = GM_d/r_d^3$. Using equations (20) and (22) gives

$$l_2l_d+m_2m_d+n_2n_d = -A\sin u+B\cos u, \tag{30}$$

and using equations (21) and (22) gives

$$\begin{aligned} l_3l_d+m_3m_d+n_3n_d &= \sin i[\cos u_d \sin(\Omega-\Omega_d)-\cos i_d \sin u_d \cos(\Omega-\Omega_d)] \\ &\qquad +\cos i \sin i_d \sin u_d \\ &= C, \text{ say.} \end{aligned} \tag{31}$$

Substituting equations (25), (30) and (31) in (29), the components of the disturbing force become:

$$\begin{aligned} S = -K\Big[&r1-\tfrac{3}{2}(A^2+B^2)-3AB\sin 2u-\tfrac{3}{2}(A^2-B^2)\cos 2u \\ &+\frac{3r}{2r_d}(A\cos u+B\sin u)\{3-5-(A\cos u+B\sin u)\}\Big] \end{aligned} \tag{32}$$

$$T = 3Kr\left[AB\cos 2u - \tfrac{1}{2}(A^2-B^2)\sin 2u) + \right.$$
$$\left. + \frac{r}{2r_d}(A\sin u - B\cos u)\{1-5(A\cos u + B\sin u)^2\}\right] \qquad (33)$$

$$W = 3KrC\left[A\cos u + B\sin u - \frac{r}{2r_d}\{1-5(A\cos u + B\sin u)^2\}\right] \qquad (34)$$

where the error due to the neglected terms is a factor

$$\left\{1 + O\left(\frac{r^2}{r_d^2}\right)\right\}.$$

The second-order terms are only needed in the expression for ω.

It is worth noting that A, B and C, defined by equations (26) and (31), are direction cosines of the radius vector to the disturbing body referred to geocentric axes through the ascending node of the satellite, through the apex of the orbit and normal to the orbit respectively.

3.2. *Change per revolution in the orbital elements*

3.2.1. *Semi-major axis*

Substituting for S and T from (32) and (33), equation (10) becomes

$$\frac{da}{d\theta} = \frac{2Kr^3}{n^2a^2(1-e^2)}\{[-1+\tfrac{3}{2}(A^2+B^2)+3AB\sin 2u + \tfrac{3}{2}(A^2-B^2)\cos 2u]e\sin\theta$$
$$+3[AB\cos 2u - \tfrac{1}{2}(A^2-B^2)\sin 2u][1+e\cos\theta]\} \qquad (35)$$

The change in the semi-major axis, Δa, during one revolution of the satellite is obtained by substituting $u = \theta+\omega$ in equation (35) and integrating over a complete revolution. A and B (and, in later equations, C) are considered constant during the integrations because the mean motion of the disturbing body is very much less than that of the satellite. Hence, noting that all integrals of the form

$$\int_0^{2\pi} \frac{\sin n\theta}{f(\cos\theta)}d\theta \quad \text{are zero, we have}$$

$$\Delta a = \frac{6K(1-e^2)^2a}{n^2}[AB\cos 2\omega - \tfrac{1}{2}(A^2-B^2)\sin 2\omega]\int_0^{2\pi}\frac{(e\cos\theta+\cos 2\theta)}{(1+e\cos\theta)^3}d\theta,$$

so that

$$\Delta a = 0, \qquad (36)$$

to the first order.

3.2.2. *Eccentricity*

From equations (11), (32) and (33)

$$\frac{de}{d\theta} = \frac{Kr^3}{n^2a^3}\Big\{[-1+\tfrac{3}{2}(A^2+B^2)+3AB\sin 2u+\tfrac{3}{2}(A^2-B^2)\cos 2u]\sin\theta$$
$$+3[AB\cos 2u - \tfrac{1}{2}(A^2-B^2)\sin 2u]\left[\cos\theta + \frac{\cos\theta+e}{1+e\cos\theta}\right]\Big\} \qquad (37)$$

Integrating over one revolution, we find

$$\Delta e = \frac{3K(1-e^2)^3}{n^2}[AB\cos 2\omega - \tfrac{1}{2}(A^2-B^2)\sin 2\omega]\int_0^{2\pi}\left\{\frac{\cos\theta}{(1+e\cos\theta)^3}+\right.$$

$$\left.+\frac{e\cos 2\theta}{(1+e\cos\theta)^4}+\frac{\cos 2\theta\cos\theta}{(1+e\cos\theta)^4}\right\}d\theta$$

$$= -\frac{15\pi Ke(1-e^2)^{\frac{1}{2}}}{n^2}[AB\cos 2\omega - \tfrac{1}{2}(A^2-B^2)\sin 2\omega]$$

$$= -\frac{5\pi e}{n^2 a}\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} T_p, \tag{38}$$

from (33), where the suffix p refers to the value at perigee.

3.2.3. *Perigee height*

The radial distance of perigee, r_p, is given by

$$r_p = a(1-e), \tag{39}$$

so that the change in perigee height during one revolution is

$$\Delta r_p = -a\Delta e. \tag{40}$$

Therefore, using (38),

$$\Delta r_p = \frac{5\pi e}{n^2}\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} T_p. \tag{41}$$

3.2.4. *Right ascension of ascending node*

Substituting equation (34) for W in (13) gives

$$\frac{d\Omega}{d\theta} = \frac{3Kr^4C}{n^2a^4(1-e^2)\sin i}[A\cos u + B\sin u]\sin u. \tag{42}$$

Integrating over one revolution,

$$\Delta\Omega = \frac{3K(1-e^2)^3C}{2n^2\sin i}\int_0^{2\pi}\frac{A\sin 2\omega\cos 2\theta + B(1-\cos 2\omega\cos 2\theta)}{(1+e\cos\theta)^4}d\theta$$

$$= \frac{3\pi KC}{2n^2(1-e^2)^{\frac{1}{2}}\sin i}[5Ae^2\sin 2\omega + B(2+3e^2-5e^2\cos 2\omega)]. \tag{43}$$

3.2.5. *Orbital inclination*

From equations (12) and (34)

$$\frac{di}{d\theta} = \frac{3Kr^4C}{n^2a^4(1-e^2)}[A\cos u + B\sin u]\cos u. \tag{44}$$

Integrating,

$$\Delta i = \frac{3K(1-e^2)^3C}{2n^2}\int_0^{2\pi}\frac{A(1+\cos 2\omega\cos 2\theta)+B\cos 2\theta\sin 2\omega}{(1+e\cos\theta)^4}d\theta$$

$$= \frac{3\pi KC}{2n^2(1-e^2)^{\frac{1}{2}}}[A(2+3e^2+5e^2\cos 2\omega)+5Be^2\sin 2\omega]. \quad (45)$$

3.2.6. *Argument of perigee*

Combining (13) and (14), substituting S and T from (32) and (33) we obtain

$$\frac{d\omega}{d\theta}+\frac{d\Omega}{d\theta}\cos i =$$

$$\frac{Kr^3}{n^2a^3e}\Big\{[1-\tfrac{3}{2}(A^2+B^2)-3AB\sin 2u-\tfrac{3}{2}(A^2-B^2)\cos 2u]\cos\theta$$

$$+3[AB\cos 2u-\tfrac{1}{2}(A^2-B^2)\sin 2u]\left[1+\frac{1}{1+e\cos\theta}\right]\sin\theta\Big\}. \quad (46)$$

Integrating,

$$\Delta\omega+\Delta\Omega\cos i =$$

$$\frac{K(1-e^2)^3}{n^2e}\int_0^{2\pi}\Big\{\frac{[1-\frac{3}{2}(A^2+B^2)]\cos\theta}{(1+e\cos\theta)^3}-$$

$$-3[AB\sin 2w+\tfrac{1}{2}(A^2-B^2)\cos 2\omega]\left[\frac{\cos\theta}{(1+e\cos\theta)^3}+\frac{\sin 2\theta\sin\theta}{(1+e\cos\theta)^4}\right]\Big\}d\theta$$

$$= \frac{3\pi K(1-e^2)^{\frac{1}{2}}}{n^2}\{5[AB\sin 2\omega+\tfrac{1}{2}(A^2-B^2)\cos 2\omega]-1+\tfrac{3}{2}(A^2+B^2)\}. \quad (47)$$

For moderately small eccentricities, second-order terms become important for the argument of perigee. If the second-order terms of equations (32) and (33) are included in (46), we obtain the term to be added to (47). It is

$$\frac{15\pi Ka(A\cos\omega+B\sin\omega)}{2r_dn^2e}\{1-\tfrac{5}{4}(A^2+B^2)\}.$$

3.3 *Rates of change of the orbital elements*

The rate of change of any orbital element averaged over a complete revolution of the satellite is readily obtained from the results of the previous section by utilizing the fact that Δt, the increment of time during one revolution, is equal to the period of revolution.

Using equation (15) in conjunction with the results of the previous section, the rates of change of the elements can be written as

$$\dot{a} = 0+O(\Gamma ae) \quad (48)$$

$$\dot{e} = -\frac{15}{2}\frac{K}{n}e(1-e^2)^{\frac{1}{2}}[AB\cos 2\omega-\tfrac{1}{2}(A^2-B^2)\sin 2\omega]+O(\Gamma) \quad (49)$$

$$\dot{r}_p = \frac{15}{2}\frac{K}{n}ae(1-e^2)^{\frac{1}{2}}[AB\cos 2\omega-\tfrac{1}{2}(A^2-B^2)\sin 2\omega]+O(\Gamma a) \quad (50)$$

$$\dot{\Omega} = \frac{3KC}{4n(1-e^2)^{\frac{1}{2}}\sin i}[5Ae^2\sin 2\omega + B(2+3e^2-5e^2\cos 2w)] + O(\Gamma) \qquad (51)$$

$$\frac{di}{dt} = \frac{3KC}{4n(1-e^2)^{\frac{1}{2}}}[A(2+3e^2+5e^2\cos 2\omega)+5Be^2\sin 2\omega] + O(\Gamma) \qquad (52)$$

$$\dot{\omega} + \dot{\Omega}\cos i = \frac{3}{2}\frac{K}{n}(1-e^2)^{\frac{1}{2}}\Big[5\{AB\sin 2\omega + \tfrac{1}{2}(A^2-B^2)\cos 2\omega\} - 1 + \tfrac{3}{2}(A^2+B^2)$$

$$\frac{5a}{2er_d}\{1-(\tfrac{5}{4}A^2+B^2)\}(A\cos\omega + B\sin w)\Big] + O\left(\frac{\Gamma a}{er_d}, \frac{\Gamma ae}{r_d}\right), \qquad (53)$$

where

$$T = \frac{Ka}{nr_d} \qquad (54)$$

3.4. *Resonance relations*

The possible existence of resonance is well known in planetary theory (Brown & Shook 1933, p. 248). When resonance occurs, the eccentricity is the most important orbital element since any change in it affects the perigee radius, which influences the satellite's lifetime. After substituting for A and B and performing considerable trigonometric manipulation, equation (49) becomes

$$\dot{e} = -\frac{15}{4}\frac{K}{n}e(1-e^2)^{\frac{1}{2}} \times$$

$$[\sin^4\tfrac{1}{2}i\{\cos^4\tfrac{1}{2}i_d\sin 2(\beta-u_d-\omega)+\sin^4\tfrac{1}{2}i_d\sin 2(\beta+u_d-\omega)+\tfrac{1}{2}\sin^2 i_d\sin 2(\beta-\omega)\} -$$

$$-\cos^4\tfrac{1}{2}i\{\cos^4\tfrac{1}{2}i_d\sin 2(\beta-u_d+\omega)+\sin^4\tfrac{1}{2}i_d\sin 2(\beta+u_d+\omega)+\tfrac{1}{2}\sin^2 i_d\sin 2(\beta+\omega)\}$$

$$+\cos^2\tfrac{1}{2}i\sin i\sin i_d\{\cos i_d\sin(\beta+2\omega)+\cos^2\tfrac{1}{2}i_d\sin(2u_d-\beta-2\omega)+\sin^2\tfrac{1}{2}i_d\times$$

$$\times\sin(\beta+2u_d+2\omega)\}+\sin^2\tfrac{1}{2}i\sin i\sin i_d\times$$

$$\times\{\cos i_d\sin(\beta-2\omega)+\cos^2\tfrac{1}{2}i_d\sin(2u_d-\beta+2\omega)+\sin^2\tfrac{1}{2}i_d\sin(\beta+2u_d-2\omega)\}$$

$$-\tfrac{3}{8}\sin^2 i\sin^2 i_d\{\sin 2(\omega+u_d)+\sin 2(\omega-u_d)\}$$

$$-\tfrac{1}{2}\sin^2 i\{1-\tfrac{3}{2}\sin^2 i_d\}\sin 2\omega], \qquad (55)$$

where $\beta = \Omega - \Omega_d$. The fifteen possible cases when resonance can occur are:

$$\left.\begin{array}{ll} \dot{\beta}\pm\dot{u}_d\pm\dot{\omega} & = 0 \\ 2\dot{u}_d\pm\dot{\beta}\pm 2\dot{\omega} & = 0 \\ \dot{\beta}\pm\dot{\omega} & = 0 \\ \dot{\beta}\pm 2\dot{\omega} & = 0 \\ \dot{\omega}\pm\dot{u}_d & = 0 \\ \text{and} \quad \dot{\omega} & = 0 \end{array}\right\} \qquad (56)$$

To determine whether resonance occurs, all perturbing influences must be included in these relations, although the Earth's oblateness will normally be the main one.

3.5. *Circular orbits*

A circular orbit undergoes no change in size or shape to the order of accuracy considered here, and Ω and i are the only elements of interest. Equations (51) and (52) become

$$\dot{\Omega} = \frac{3}{2}\frac{K}{n}\frac{BC}{\sin i} \tag{57}$$

and

$$\frac{di}{dt} = \frac{3}{2}\frac{K}{n}AC. \tag{58}$$

4. Perturbations due to solar radiation pressure

4.1. *Force on a satellite due to solar radiation*

As the Earth's distance from the Sun is large compared with the size of the orbit, the force produced on a satellite by solar radiation pressure can be assumed independent of its distance from the Sun. Also, in a first approximation, it is possible to neglect the effect of the Earth's albedo. For a non-spherical satellite the magnitude of the force will depend on the satellite's orientation, but for the purpose of this analysis we assume that it is possible to use a suitable average value, F per unit mass, which acts while the satellite is in sunlight.

Using the notation of Section 3 the force can be resolved into components:

$$S = F(A\cos u + B\sin u) \tag{59}$$

$$T = F(-A\sin u + B\cos u) \tag{60}$$

$$W = FC. \tag{61}$$

It should be noted that the sign of F is always negative.

4.2. *Change per revolution in the orbital elements*

4.2.1. *Semi-major axis*

Substituting for S and T from equations (59) and (60), equation (10) becomes

$$\frac{da}{d\theta} = -\frac{2F(1-e^2)}{n^2}\left[\frac{A\sin u - B\cos u + e(A\sin\omega - B\cos\omega)}{(1+e\cos\theta)^2}\right].$$

If the values of true anomaly when the satellite departs from and enters the Earth's shadow are denoted by θ_0 and θ_c respectively, the change in a during one revolution is given by

$$\Delta a = -\frac{2F(1-e^2)}{n^2}\int_{\theta_0}^{\theta_c}\frac{(A\sin\omega - B\cos\omega)(e+\cos\theta)+(A\cos\omega+B\sin\omega)\sin\theta}{(1+e\cos\theta)^2}d\theta.$$

Evaluating the integral and using the equation of the orbit, we obtain

$$\Delta a = -\frac{2F}{n^2 a}\left[(A\sin\omega - B\cos\omega)(r_c\sin\theta_c - r_0\sin\theta_0)\right.$$
$$\left. + (A\cos\omega + B\sin\omega)\left(\frac{r_c - r_0}{e}\right)\right]. \tag{62}$$

The result may be written in terms of the components of the perturbing force at perigee, denoted by the suffix p, evaluated as if perigee is in sunlight:

$$\Delta a = \frac{2}{n^2 a}[(r_c \sin\theta_c - r_0 \sin\theta_0)T_p + a(\cos E_c - \cos E_0)S_p]. \quad (63)$$

Alternatively, if the distance travelled by the satellite, while in sunlight, in the direction of the Sun is h,

$$\Delta a = \frac{2Fh}{n^2 a}. \quad (64)$$

4.2.2. *Eccentricity*

From equations (11), (59) and (60), we have

$$\frac{de}{d\theta} = -\frac{Fr^2}{\mu}\left[(A\sin\omega - B\cos\omega) + (A\sin u - B\cos u)\left(\frac{\cos\theta + e}{1 + e\cos\theta}\right)\right]$$

where $\mu = n^2a^3$. Integrating,

$$\Delta e = \frac{T_p}{\mu}\left[3a^2(1-e^2)^{\frac{1}{2}}\left\{\tan^{-1}\left(\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_c)}{1+e}\right) - \tan^{-1}\left(\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_0)}{1+e}\right)\right\}\right.$$
$$\left. - \frac{1}{2e}(r_c^2\sin\theta_c - r_0^2\sin\theta_0) + \frac{a}{2e}(1-4e^2)(r_c\sin\theta_c - r_0\sin\theta_0)\right]$$
$$- \frac{S_p}{2\mu}\left[(r_c^2 - r_0^2) + \frac{a(1-e^2)}{e^2}(r_c - r_0) + \frac{1}{e}(r_c^2\cos\theta_c - r_0^2\cos\theta_0)\right]. \quad (65)$$

4.2.3. *Perigee height*

The change in perigee radius can be obtained from

$$\Delta r_p = (1-e)\,\Delta a - a\Delta e. \quad (66)$$

4.2.4. *Right ascension of ascending node*

Using equation (13) and noting that W is constant round the orbit, we have

$$\frac{d\Omega}{d\theta} = \frac{Wa^2(1-e^2)^2}{\mu\sin i}\frac{\sin(\theta+\omega)}{(1+e\cos\theta)^3}.$$

Integrating,

$$\Delta\Omega =$$
$$\frac{W}{\mu\sin i}\left\{\frac{(r_c^2 - r_0^2)}{2e}\cos\omega + \left[\frac{r_c^2\sin\theta_c - r_0^2\sin\theta_0}{2(1-e^2)} + \frac{(1+2e^2)}{2(1-e^2)}(ar_c\sin\theta_c - ar_0\sin\theta_0)\right.\right.$$
$$\left.\left. - \frac{3a^2e}{(1-e^2)^{\frac{1}{2}}}\left(\tan^{-1}\left\{\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_c)}{1+e}\right\} - \tan^{-1}\left\{\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_0)}{1+e}\right\}\right)\right]\sin\omega\right\}. \quad (67)$$

4.2.5. *Orbital inclination*

From equation (12)

$$\frac{di}{d\theta} = \frac{Wa^2(1-e^2)^2}{\mu}\frac{\cos(\theta+\omega)}{(1+e\cos\theta)^3}.$$

Integrating,

$$\Delta i = \frac{W}{\mu}\left\{\left[\frac{r_c^2\sin\theta_c - r_0^2\sin\theta_0}{2(1-e^2)} + \frac{(1+2e^2)}{2(1-e^2)}(ar_c\sin\theta_c - ar_0\sin\theta_0)\right.\right.$$
$$\left. - \frac{3a^2e}{(1-e^2)^{\frac{1}{2}}}\left(\tan^{-1}\left\{\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_c)}{1+e}\right\} - \tan^{-1}\left\{\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_0)}{1+e}\right\}\right)\right]\cos\omega$$
$$\left. - \frac{(r_c^2-r_0^2)}{2e}\sin\omega\right\}. \tag{68}$$

4.2.6. *Argument of perigee*

Combining equations (13), (14), (59) and (60) we obtain

$$\frac{d\omega}{d\theta} + \frac{d\Omega}{d\theta}\cos i = -\frac{Fr^2}{\mu e}\left\{(A\cos\omega + B\sin\omega) + (A\sin u - B\cos u)\frac{\sin\theta}{1+e\cos\theta}\right\}.$$

Integrating,

$$\Delta\omega + \Delta\Omega\cos i =$$
$$-\frac{S_p}{\mu e}\left[3a^2(1-e^2)^{\frac{1}{2}}\left(\tan^{-1}\left\{\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_c)}{1+e}\right\} - \tan^{-1}\left\{\frac{(1-e^2)^{\frac{1}{2}}\tan(\frac{1}{2}\theta_0)}{1+e}\right\}\right)\right.$$
$$\left. + \frac{1}{2e}(r_c^2\sin\theta_c - r_0^2\sin\theta_0) - \frac{a}{2e}(1+2e^2)(r_c\sin\theta_c - r_0\sin\theta_0)\right]$$
$$+ \frac{T_p}{2\mu e^3}[e(r_c^2\cos\theta_c - r_0^2\cos\theta_0) + a(1-e^2)(r_c - r_0)]. \tag{69}$$

4.3. *Rates of change of the orbital elements*

The rate of change of any orbital element, ψ, averaged over one revolution of the satellite is easily obtained by combining equation (15) with the appropriate equation of Section 4.2. The result, which is exact if F is constant, is

$$\dot{\psi} = \frac{n\Delta\psi}{2\pi}. \tag{70}$$

4.4. *Results for a satellite permanently in sunlight*

For a satellite permanently in sunlight the integral is taken from 0 to 2π and the above results simplify considerably:

$$\dot{a} = 0 \tag{71}$$

$$\dot{e} = \frac{3(1-e^2)^{\frac{1}{2}}}{2na}T_p \tag{72}$$

$$\dot{r}_p = -a\,\dot{e} \tag{73}$$

$$\dot{\Omega} = -\frac{3We\sin\omega}{2na(1-e^2)^{\frac{1}{2}}\sin i} \tag{74}$$

$$\frac{di}{dt} = -\frac{3We\cos\omega}{2na(1-e^2)^{\frac{1}{2}}} \tag{75}$$

$$\dot{\omega} + \dot{\Omega}\cos i = -\frac{3(1-e^2)^{\frac{1}{2}}}{2nae}S_p \tag{76}$$

The components of the perturbing force appearing in these equations are obtained by substituting for A, B and C from equations (26) and (31) in equations (59) to (61). Writing $i_d = \epsilon$, $\Omega_d = 0$ and $u_d = L$ for the Sun (see Section 5), we obtain:

$$S_p = F\{[\cos^2\tfrac{1}{2}\epsilon\cos(\omega+\Omega-L)+\sin^2\tfrac{1}{2}\epsilon\cos(\omega+\Omega+L)]\cos^2\tfrac{1}{2}i + [\cos^2\tfrac{1}{2}\epsilon\cos(\omega-\Omega+L)+\sin^2\tfrac{1}{2}\epsilon\cos(\omega-\Omega-L)]\sin^2\tfrac{1}{2}i + \tfrac{1}{2}[\cos(\omega-L)-\cos(\omega+L)]\sin i\sin\epsilon\} \tag{77}$$

$$T_p = -F\{[\cos^2\tfrac{1}{2}\epsilon\sin(\omega+\Omega-L)+\sin^2\tfrac{1}{2}\epsilon\sin(\omega+\Omega+L)]\cos^2\tfrac{1}{2}i + [\cos^2\tfrac{1}{2}\epsilon\sin(\omega-\Omega+L)+\sin^2\tfrac{1}{2}\epsilon\sin(\omega-\Omega-L)]\sin^2\tfrac{1}{2}i - \tfrac{1}{2}[\sin(\omega+L)-\sin(\omega-L)]\sin i\sin\epsilon\} \tag{78}$$

$$W\sin\omega = -\frac{F}{2}\{[\cos(\omega+\Omega-L)-\cos(\omega-\Omega+L)]\sin i\cos^2\tfrac{1}{2}\epsilon + [\cos(\omega+\Omega+L)-\cos(\omega-\Omega-L)]\sin i\sin^2\tfrac{1}{2}\epsilon + [\cos(\omega+L)-\cos(\omega-L)]\cos i\sin\epsilon\} \tag{79}$$

$$W\cos\omega = \frac{F}{2}\{[\sin(\omega+\Omega-L)-\sin(\omega-\Omega+L)]\sin i\cos^2\tfrac{1}{2}\epsilon + [\sin(\omega+\Omega+L)-\sin(\omega-\Omega-L)]\sin i\sin^2\tfrac{1}{2}\epsilon + [\sin(\omega+L)-\sin(\omega-L)]\cos i\sin\epsilon\}. \tag{80}$$

When the orbit is entirely in sunlight, there are six conditions for resonance, which are easily recognized. They are given by

$$\left.\begin{aligned} \dot\omega\pm\dot\Omega\pm\dot L &= 0 \\ \text{and}\qquad \dot\omega\pm\dot L &= 0 \end{aligned}\right\} \tag{81}$$

These conditions are the same as six of (56).

5. Orbital elements of the Sun and the Moon

The orbital elements of the Sun and the Moon, with the ecliptic as reference plane, can be obtained from the Astronomical Ephemeris. For present purposes, sufficient accuracy is obtained by taking the orbits of the disturbing bodies as circular; r_d is then replaced by the semi-major axis, a_d, of the disturbing body, since

$$r_d = a_d\{1+0(e_d)\} \tag{82}$$

and e_d, the eccentricity of the orbit of the Sun or Moon relative to the Earth, is small. The position of the Sun is now defined by two elements and the position of the Moon by three elements. For the Sun, these are:

L, the geometric mean longitude measured in the ecliptic from the equinox of date;

ϵ the mean obliquity of the ecliptic.

For the Moon, they are:

- ☾ the mean longitude, measured in the ecliptic from the mean equinox of date to the mean ascending node of the orbit, and then along the orbit;
- ☊ the longitude of the mean ascending node of the lunar orbit on the ecliptic, measured from the mean equinox of date;
- α the inclination of the lunar orbit to the ecliptic.

Two of these elements are constants:

$$\alpha = 5{\cdot}145^\circ$$

$$\epsilon = 23{\cdot}44^\circ$$

In the notation of this paper the Sun's position is given by

$$\Omega_d = 0, \quad u_d = L, \quad i_d = \epsilon.$$

The orbital plane of the moon rotates once in 18·6 years and the value of ☊ can be obtained from the relation

$$☊ = 178{\cdot}78 - 0{\cdot}05295\, t \text{ deg},$$

where t is measured in days from 1960 January 1.0. The value of ☊ on January 1·0 is given in Table 1 for the next ten years. If the equations of spherical

Table 1

Longitude of ascending node of lunar orbit on the ecliptic

Year	☊ on January 1.0, deg
1960	178·78
1961	159·40
1962	140·07
1963	120·74
1964	101·36
1965	82·03
1966	62·70
1967	43·37
1968	23·99
1969	4·67
1970	−14·66

trigonometry are applied to the triangle in Figure 2, the Moon's position is found to be given, in the notation of this paper, by

$$\cos i_d = \cos\epsilon\cos\alpha - \sin\epsilon\sin\alpha\cos ☊$$

$$\sin\Omega_d = \frac{\sin\alpha\sin ☊}{\sin i_d}$$

and

$$u_d = ☾ - ☊ + \sin^{-1}\left\{\frac{\sin\epsilon\sin ☊}{\sin i_d}\right\}.$$

The inclination of the lunar orbit to the equator, $i_☾$, varies between 18·3° and 28·6° and $\Omega_☾$ oscillates between +13° and −13°. ☾ can be found on page 51 of the Astronomical Ephemeris.

Using equation (82) we can write

$$K = \frac{Gm_☾}{r_☾{}^3} \simeq m\bar{n}^2,$$

where, if lunar perturbations are required, m is the ratio of the mass of the Moon to the mass of the Earth and $\bar{n}$ is the mean angular motion of the Moon about the Earth. If solar perturbations are required m is taken as unity and $\bar{n}$ is the mean angular motion of the Earth about the Sun. Taking (Baker & Makemson 1960, p. 94) $m = 1/81{\cdot}45$ and the period of revolution of the Moon as 27·322 days, the value of K is 2·132 deg²/day² for lunar perturbations. For solar perturbations, the value of K is 0·9714 deg²/day².

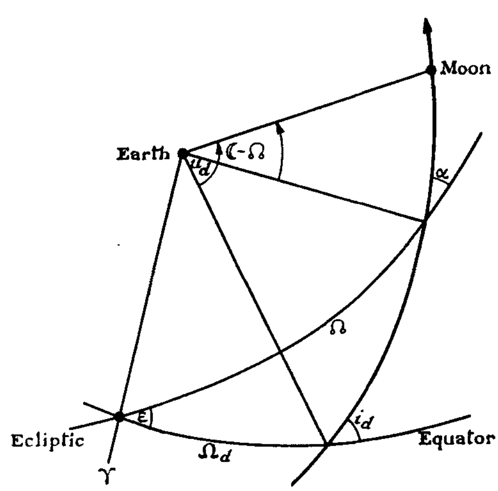

FIG. 2.—Orbital elements of the Moon.

Using a value of 1·95 cal/cm² min for the solar constant (Hynek 1951, p. 269), the solar radiation pressure on a perfectly black surface normal to the incident beam is $9{\cdot}48 \times 10^{-8}$ lb/ft² ($4{\cdot}5 \times 10^{-5}$ dyn/cm²). The total force on a curved black surface is obtained by multiplying this value by the projected area of the surface normal to the incident radiation. For total reflexion, the force will be twice that for a perfect absorber. In practice it is often difficult to decide on the exact value of the force, but its practical evaluation is outside the scope of this paper.

6. Discussion

To the first order, neither gravitation nor solar radiation pressure, when the orbit is entirely in sunlight, have any effect on the semi-major axis, so that no energy is imparted to the satellite. It should be noted that a circular orbit remains circular under the influence of the gravitational attraction of a third body, but becomes elliptic under the action of solar radiation pressure. This occurs because gravitation acts on both the Earth and the satellite, but, due to the large mass/area ratio of the Earth, solar radiation pressure only affects the satellite. In both cases, there is no change in the eccentricity when $T_p = 0$, so that the lifetime is unaffected if perigee or apogee is on the same meridian as the Sun or Moon.

Provided that the semi-major axis does not exceed one tenth of the Moon's distance from the Earth, the maximum possible value of the neglected term in equation (48) is 0·02 n.m/day; it would usually be much less, however. The maximum errors in the rates of change of the other elements, due to the neglect of high-order terms in r/r_d in equations (50) to (53) are, at most, of the order of 5 per cent. Moe (1960) has shown that the error introduced by neglecting the motion of the disturbing body during one revolution of the satellite is a factor $\{1+0(\bar{n}/n\}$. The maximum error introduced by this factor is of the same order as the other errors.

If the orbital plane of the disturbing body is taken as the reference plane, that is $i_d = 0$, the results of Section 3.2 reduce to those given by Moe (1960). In many of the references there has been a tendency to seek out only secular terms and to omit the periodic terms. If the latter are of long period, however, the amplitude of the oscillations produced by them may be large. Also resonance may occur. For most purposes only the effect of the Earth's gravitational field need be considered when investigating resonance, so that we can take (King-Hele 1958)

$$\dot{\Omega} = -10{\cdot}0\left(\frac{R_E}{a}\right)^{3\cdot5}(1-e^2)^{-2}\cos i \text{ deg/day} \tag{85}$$

$$\dot{\omega} = 5{\cdot}0\left(\frac{R_E}{a}\right)^{3\cdot5}(1-e^2)^{-2}(5\cos^2 i-1)\text{deg/day}, \tag{86}$$

where R_E is the equatorial radius of the earth, and assume that $\dot{\Omega}_d = 0$ for the Moon. Five of the relations (56) produce resonance for both lunar and solar perturbations if the orbital inclination takes one of the values given in Table 2. For lunar perturbations, five cases can never occur and the remaining five are only possible in limited ranges of the inclination. Values of $a(1-e^2)^{4/7}/R_E$ which give resonance for particular values of i are plotted in Figure 3. For solar perturbations, values of $a(1-e^2)^{4/7}/R_E$ are given in Figure 4.

Table 2

Inclinations giving resonance in the gravitational perturbations

No.	Relation	Inclinations giving resonance
1	$\dot{\beta}+\dot{\omega}=0$	46°·4 or 106°·8
2	$\dot{\beta}-\dot{\omega}=0$	73°·2 or 133°·6
3	$\dot{\beta}+2\dot{\omega}=0$	56°·1 or 111°
4	$\beta-2\dot{\omega}=0$	69° or 123°·9
5	$\dot{\omega}=0$	63°·4 or 116°·6

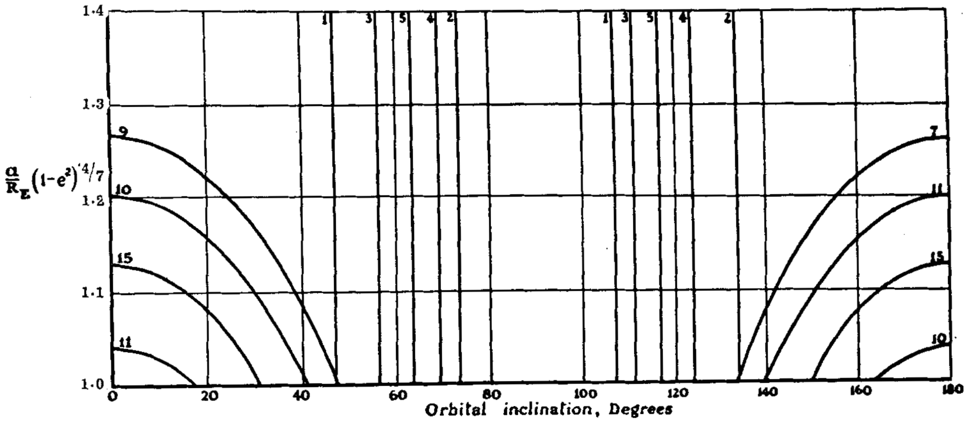

FIG. 3.—Resonant orbits for lunar perturbations.

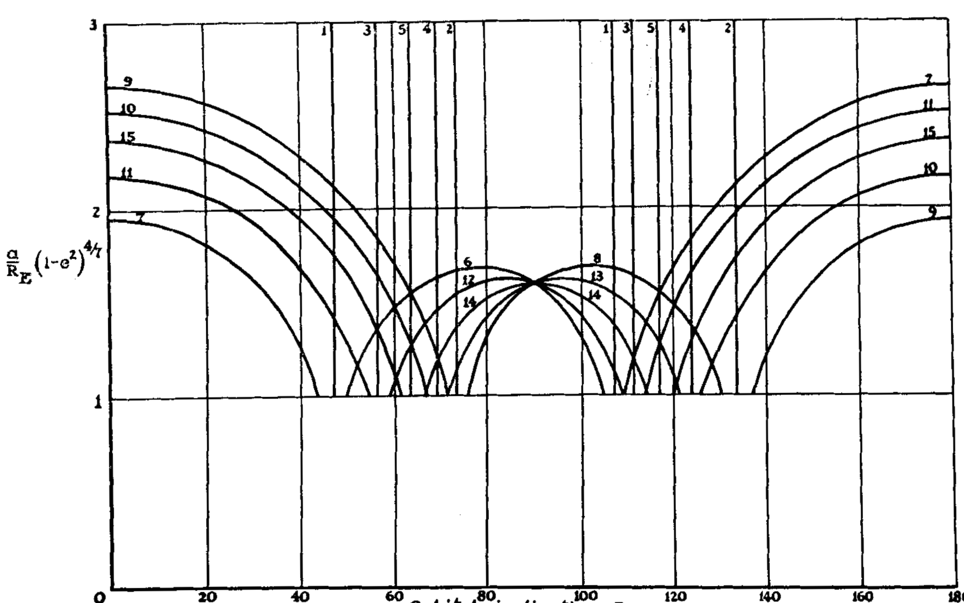

FIG. 4.—Resonant orbits for solar gravitational perturbations.

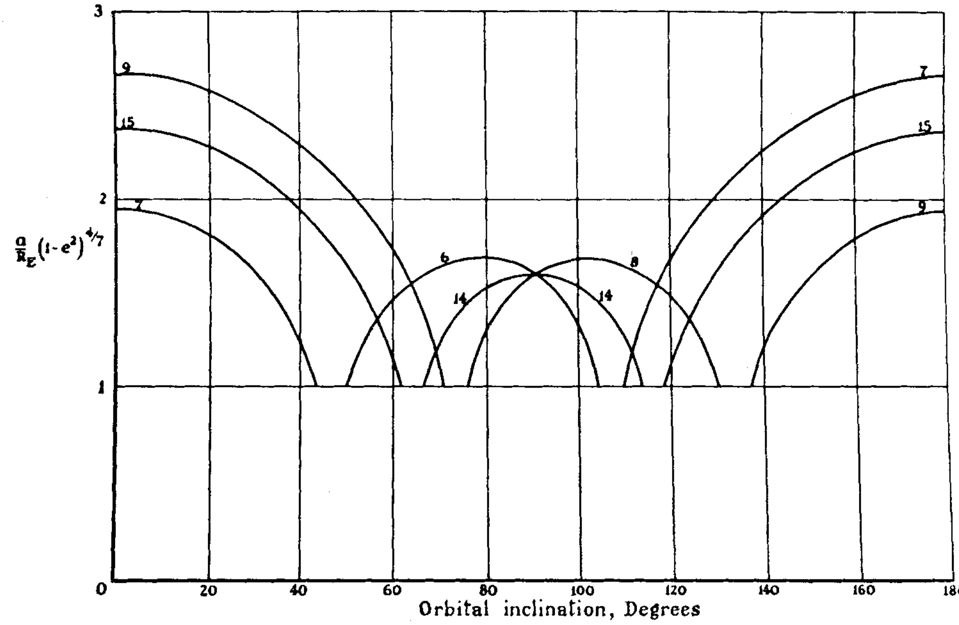

FIG. 5.—Resonant orbits for solar radiation pressure.

Table 3

Key to resonance relations plotted in Figures 3 to 5

No.	Resonance relation
6	$\dot{\omega}+\dot{\Omega}+\dot{u}_d = 0$
7	$\dot{\omega}+\dot{\Omega}-\dot{u}_d = 0$
8	$\dot{\omega}-\dot{\Omega}+\dot{u}_d = 0$
9	$\dot{\omega}-\dot{\Omega}-\dot{u}_d = 0$
10	$2\dot{u}_d+\dot{\Omega}-2\dot{\omega} = 0$
11	$2\dot{u}_d-\dot{\Omega}-2\dot{\omega} = 0$
12	$2\dot{u}_d+\dot{\Omega}+2\dot{\omega} = 0$
13	$2\dot{u}_d-\dot{\Omega}+2\dot{\omega} = 0$
14	$\dot{\omega}+\dot{u}_d = 0$
15	$\dot{\omega}-\dot{u}_d = 0$

The present theory breaks down for an equatorial orbit as the ascending node cannot be defined. This difficulty can be overcome by changing the reference plane. The ecliptic immediately suggests itself, so that $i_d = 0$ and $\Omega_d = 0$ for the Sun, and $i_d = \alpha$ and $\Omega_d =$ ☊ for the Moon.

The results for the effect of solar radiation pressure are rather lengthy; but, for many purposes, it may be possible to use the simpler results obtained by neglecting the Earth's shadow, as was done (Musen, Bryant & Bailie, 1960) when investigating the perturbations of satellite 1958 $\beta 2$ (Vanguard 1). When the orbit is entirely in sunlight resonance may occur if the value of $a(1-e^2)^{4/7}/R_E$ lies on one of the curves given in Figure 5.

7. Conclusions

The equations given in Section 3.2 of this paper enable luni-solar gravitational perturbations to be evaluated. The theory applies primarily to satellites whose semi-major axis does not exceed one tenth of the Moon's distance. At greater distances the error terms become larger, though they may still be small enough to be ignored for many purposes. The corresponding rates of change of the orbital elements averaged over one revolution can be evaluated from the equations given in Section 3.3. The results for a circular orbit are given in Section 3.5. All the necessary numerical information for evaluating the perturbations on a satellite of known size, shape and surface characteristics is given in Section 5.

The variations in the orbital elements of a particular satellite are not obvious at first sight, except for the semi-major axis. Secular variations occur in the right ascension of the ascending node and in the argument of perigee. All elements except the semi-major axis exhibit long-period variations associated with the rotation of the major axis, with the motion of the disturbing body, and with the motion of the ascending node of the satellite relative to the ascending node of the disturbing body. For a near Earth satellite, however, whose orbit decays under the influence of air drag, the long-period effects may take on the appearance of secular variations. In practice, too, the occurrence of one of the fifteen possible cases of resonance given in Section 3.4 should not be overlooked. Near-resonance is as important as resonance and the probability of this occurring by chance is quite serious, as Figure 4 shows.

The results given in Sections 4.2, 4.3 and 4.4 enable the effects of solar radiation pressure to be evaluated. Except for the semi-major axis the elements normally exhibit long-period variations when the Earth's shadow is neglected; but there are six cases of resonance, when the changes in the orbit over a long period may be large.

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