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Research note

A decomposition theorem for the transformation of local physical frames with geodetic interest

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Summary. In the present note we express the 'natural' covariant and contravariant moving frames in the actual gravity field of the Earth in terms of a 'normal' and a 'disturbing' part, the latter being expressed by the components of the deflection of the vertical and of the disturbing potential at P.

An elementary displacement of a point P in the actual gravity field of the Earth is expressed by the differential relation

$$dP = \left(\frac{\partial P}{\partial x^*}\right)^T dx^* = \mathbf{v}^{*T}\, dx^* \tag{1}$$

where

$$\left(\frac{\partial P}{\partial x^*}\right) = \left(\frac{\partial P}{\partial x^{*1}}\ \frac{\partial P}{\partial x^{*2}}\ \frac{\partial P}{\partial x^{*3}}\right)^T = \mathbf{v}^* = (\mathbf{v}_1^*\,\mathbf{v}_2^*\,\mathbf{v}_3^*)^T; \quad x^* = (x^{*1}x^{*2}x^{*3})^T,$$

x^* being the natural coordinates (see Marussi 1949), $x^{*1} = \Phi$ (astronomic latitude), $x^{*2} = \Lambda$ (astronomic longitude), $x^{*3} = W$ (actual geopotential) and $\mathbf{v}^*$ the local covariant natural frame giving, in general, the displacement reference frame; $\mathbf{v}_1^*$ directed towards the actual meridian, $\mathbf{v}_2^*$ towards the actual parallel and $\mathbf{v}_3^*$ towards the actual isozenithal. Here the term 'actual' has the meaning of the term 'geodetic' used by Marussi 1952.

Decomposing $\mathbf{v}^*$ and x^* in a normal and a disturbing part

$$\mathbf{v}^* = \mathbf{v} + \delta\mathbf{v} \tag{2}$$

$$x^* = x + \delta x \tag{3}$$

where, and from now on, the terms without asterisks are the normal part, referred to a uniquely defined model field, e.g. the Somigliana–Pizzetti ellipsoidal type of normal field, and the terms with δ the disturbing part; (1) is written

$$dP = (\mathbf{v} + \delta\mathbf{v})^T (dx + d\delta x)$$

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or

$$dP = \mathbf{v}^T dx + (\mathbf{v}^T d\delta x + \delta \mathbf{v}^T dx + \delta \mathbf{v}^T d\delta x). \tag{4}$$

Obviously $x = (x^1 x^2 x^3)^T$ are the normal coordinates at P, $x^1 = \phi$, $x^2 = \lambda$, $x^3 = U$ and $\delta x = (\delta x^1 \delta x^2 \delta x^3)^T$ are the coordinate-disturbances, where $\delta x^1 = \xi$, $\delta x^2 = \epsilon = \eta/\cos\phi$, (the deflections of the vertical) and $\delta x^3 = T$, (the disturbing potential) at P.

Denoting by $\mathbf{u}^* = (\mathbf{u}_1^* \mathbf{u}_2^* \mathbf{u}_3^*)^T$ the local contravariant natural frame, which describes gradient vectors, we have by definition, (see, e.g. Marussi 1976, p. 7)

$$\mathbf{u}^* = \operatorname{grad} x^* \tag{5}$$

with

$$\mathbf{u}^* \mathbf{v}^{*T} = I \tag{6}$$

where I the 3×3 unit matrix.

Decomposing again $\mathbf{u}^*$ and x^*, within (5), in a normal and a disturbing part, we obtain

$$\mathbf{u}^* = \mathbf{u} + \delta\mathbf{u} \tag{7}$$

and introducing (7) and (3) into (5)

$$\mathbf{u}^* = \operatorname{grad} x + \operatorname{grad} \delta x \tag{8}$$

where obviously

$$\mathbf{u} = \operatorname{grad} x \tag{9}$$

and

$$\delta\mathbf{u} = \operatorname{grad} \delta x. \tag{10}$$

Introducing the astronomic orthonormal base $\mathbf{i}^* = (\mathbf{i}_1^* \mathbf{i}_2^* \mathbf{i}_3^*)^T$ at P, where $\mathbf{i}_1^*$ is directed towards the actual north, $\mathbf{i}_2^*$ towards the actual east and $\mathbf{i}_3^*$ towards the actual zenith, the triad being clockwise, we could express $\mathbf{u}$ by means of $\mathbf{i}^*$ through a 3×3 matrix $(A + \delta A)$

$$\mathbf{u} = (A + \delta A)\, \mathbf{i}^* \tag{11}$$

where A the normal part of the matrix and δA the disturbing part. The normal part A, concerning a Somigliana–Pizzetti type of normal field, is given by Marussi (1952, p. 83, equation V-27, or: 1976, p. 30, equation 4.3) referred to the normal orthonormal base $\mathbf{i} = (\mathbf{i}_1 \mathbf{i}_2 \mathbf{i}_3)^T$ at P, where $\mathbf{i}_1$ is directed towards the normal north, $\mathbf{i}_2$ towards the normal east and $\mathbf{i}_3$ towards the normal zenith, the triad being clockwise. The normal part A could be also derived for any other model field, for instance the SAO satellite gravity model. The disturbing part δA relates the $\mathbf{i}$ and the $\mathbf{i}^*$ frames through rotations involving the deflection components δx^1 and δx^2 and their partial derivatives, (see, e.g. Livieratos 1976, appendix 1.)

Furthermore we can express the gradients of the scalar quantities δx by means of $\mathbf{u}$ through a 3×3 Jacobian matrix B

$$\operatorname{grad} \delta x = B\mathbf{u} \tag{12}$$

(see, e.g. Finzi & Pastori 1971, p. 21, section 4). The Jacobian matrix B involves the partial derivatives of δx with respect to the normal coordinates x. Introducing (11) into (12) we obtain

$$\operatorname{grad} \delta x = B(A + \delta A)\, \mathbf{i}^*. \tag{13}$$

Replacing (11) and (13) into (7) we can write, with the help of (8)

$$\mathbf{u}^* = (A + \delta A)\, \mathbf{i}^* + B(A + \delta A)\, \mathbf{i}^* \tag{14}$$

or

$$\mathbf{u}^* = (A + BA + \delta A + B\delta A)\,\mathbf{i}^*. \tag{15}$$

Using the condition (6), and having by definition

$$\mathbf{i}^*\mathbf{i}^{*T} = I \tag{16}$$

we can write

$$\mathbf{v}^* = [(A + BA + \delta A + B\delta A)^T]^{-1}\,\mathbf{i}^* \tag{17}$$

if and only if $\det(A + BA + \delta A + B\delta A) \neq 0$.

The transformation matrices in (15) and (17) relate the covariant and the contravariant local frames with respect to the astronomic orthonormal base $\mathbf{i}^*$ at P, and depend only upon three simple 3×3 matrices A, B and δA, the matrix A being the normal part and the matrix $(BA + \delta A + B\delta A)$ the disturbing part.

Having already established the local covariant vector frame $\mathbf{v}^*$, it is easy to derive the metric tensor M^* in the actual gravity field, defined by the relation

$$M^* = \mathbf{v}^*\mathbf{v}^{*T}. \tag{18}$$

Introducing (17) into (18) we obtain

$$M^* = [(A + BA + \delta A + B\delta A)^T]^{-1}\,\mathbf{i}^*\,\mathbf{i}^{*T}(A + BA + \delta A + B\delta A)^{-1} \tag{19}$$

and recalling (16)

$$M^* = [(A + BA + \delta A + B\delta A)\,(A + BA + \delta A + B\delta A)^T]^{-1}. \tag{20}$$

Performing matrix algebra, we obtain the metric tensor decomposed into a normal and a disturbing part

$$\begin{aligned} M^* = (&AA^T + \\ &+ AA^TB^T + A\delta A^T + BAA^T + \delta AA^T + \\ &+ A\delta A^TB^T + BA\delta A^T + \delta AA^TB^T + \delta A\delta A^T + B\delta AA^T + BAA^TB^T + \\ &+ BA\delta A^TB^T + \delta A\delta A^TB^T + B\delta AA^TB^T + B\delta A\delta A^T + \\ &+ B\delta A\delta A^TB^T)^{-1}. \end{aligned} \tag{21}$$

The reciprocal components of the metric could be as well derived from the definition $\mathbf{u}^*\mathbf{u}^{*T}$.

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