An Integral Equation and its Solution for some Two- and Three-Dimensional Problems in Resistivity and Induced Polarization

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Summary

The potential, $U$, about a point electrode, at the surface of a layered ground in which there is an heterogeneity embedded, satisfies the integral equation:

$$U = U_* + \frac{2(\sigma_* - \sigma)}{(\sigma_* + \sigma)} \int \frac{\partial U}{\partial n} G ds.$$

Here, $U_*$ and $\sigma_*$ are the corresponding quantities for the potential and conductivity without the heterogeneity. The integral is taken over the surface of the heterogeneity, $\partial U/\partial n$ is the normal derivative (in the direction of the outward normal) of $U$, and $G$ is a Green's function.

Solutions to this equation can readily be found by using the Galerkin method of solving integral equations. The solutions of this equation when the heterogeneity is a sphere or a cylinder in a uniform ground or beneath a conductive overburden are the most readily found.

When the solution of the integral has been found for the potential it is a simple matter to calculate the apparent resistivity or chargeability for any electrode configuration.

Introduction

The practical geophysicist has often had to employ analogue and numerical model studies to help him interpret the scalar or vector fields which he has measured in the field in terms of geology. The result of this diversion has been to build up an extensive literature relating to the behaviour of various models, which very often only poorly model the actual field geology.

With the advent of electronic computers, however, the geological models have become very much more closely related to the actual geological structures. A serious shortcoming, however, has been that despite the availability of computers the number of algorithms available to the geophysicist in order to solve the field equations is very few. Where the geometry is very simple the theory of boundary value problems may be used to solve the appropriate differential equations for the required field quantities. For two-dimensional structures the unknown field quantity can be considered to be a scalar quantity and techniques such as the finite element method (Coggon 1971) or the network theory (Swift 1967) have been used to obtain useful solutions.

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Table 1

Some useful formulae

(1)

\[ \int_0^\infty F(\lambda) J_0(\lambda r) \, d\lambda = \frac{1}{2\pi} \int_0^{2\pi} \frac{F(\lambda)}{\lambda} e^{i\alpha x + i\beta y} \, d\alpha d\beta \]

where

\[ r = \sqrt{(x^2 + y^2)} \]
\[ \lambda = \sqrt{(x^2 + y^2)} \]

Tranter (1966) p. 12

(2)

\[ \int_0^{2\pi} \int_0^\infty \frac{r \, dr \, d\theta}{(r^2 + d^2)^{3/2}} \left( r^2 + \rho^2 - 2\rho r \cos \theta + (z + \beta)^2 \right)^{1/2} = \frac{2\pi}{\sqrt{(\rho^2 + (d + (z + \beta))^2)}}, \]

Stevenson (1934) p. 124

(3) The potential function \( U_1 \) for a uniform half space with the geometry shown in Fig. 1(a) is

\[ U_1 = \frac{I_0}{2\pi} \int_0^\infty J_0(\lambda r) \, d\lambda = \frac{I_0}{2\pi \sqrt{(r^2 + z^2)}} \]

\[ r^2 = x^2 + y^2 \]

Stefanesco et al. (1930)

(4) The potential functions \( U_{12}, U_{22} \) for a two-layered ground with the geometry shown in Fig. 1(c) is:

\[ U_{12} = \frac{I_0}{2\pi} \int_0^\infty J_0(\lambda r) \left[ e^{-\lambda x} + \frac{k_1 e^{-2\lambda t}}{1 - k_1 e^{-2\lambda t}} \right] \left[ e^{-\lambda x} + e^{\lambda x} \right] \, d\lambda \]

\[ U_{22} = \frac{I_0}{2\pi} \left[ 1 + k_1 \right] \int_0^\infty \frac{e^{-\lambda x}}{(1 - k_1 e^{-2\lambda t})} J_0(\lambda r) \, d\lambda \]

where

\[ k_1 = (\rho_2 - \rho_1) / (\rho_1 + \rho_2), \quad r = \sqrt{(x^2 + z^2)} \]

Stefanesco et al. (1930)

(5) The Green's function, \( G \) satisfies

\[ \nabla^2 G = -\delta(x - x_1) \delta(y - y_1) \delta(z - z_1) \]

in each sub region of Fig. 1(a) and (c) and is subject to the same boundary conditions as the electric potential for a current source at \((x_1, y_1, z_1)\). Therefore for a layered medium it can be found in an analogous manner to that employed by Stefanesco et al. (1930) to find the potential in a layered medium. For a half space of geometry 1(a) then:

\[ G_1 = \frac{1}{4\pi} \left[ \frac{1}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} \right] \]

For a two-layered medium as in Fig. 1(c):

\[ G_1 = \frac{1}{4\pi} \int_0^\infty J_0(\lambda s) \left[ e^{-\lambda (x + z_1)} + e^{-\lambda (x - z_1)} \right] \left( \frac{1 - k_1}{1 - k_1 e^{-2\lambda t}} \right) \, d\lambda \]

and

\[ G_2 = \frac{1}{4\pi} \int_0^\infty J_0(\lambda s) \left[ e^{-\lambda (x - z_1)} + \frac{e^{-\lambda (x + z_1)} (1 - k_1 e^{2\lambda t})}{1 - k_1 e^{-2\lambda t}} \right] \, d\lambda \]

where

\[ k_1 = (\rho_2 - \rho_1) / (\rho_1 + \rho_2) \]

and

\[ s = \sqrt{(x-x_1)^2 + (y-y_1)^2} \]
Resistivity and induced polarization

\[ P_n^m(\cos \theta) = \frac{(-i)^n (n+m)!}{2n!} \int_0^{\pi} (\cos \theta + i \sin \theta \cos \phi)^n \cos \phi m \phi d\phi \]

where

\[ P_n^m(\cos \theta) = \sin^n \theta \frac{d^m}{d(\cos \theta)^m} P_n(\cos \theta) \]

is the Legendre polynomial of degree \( n \).

Whittaker (1902) p. 338

\[ \int_0^{\infty} \frac{\cos xy \, dy}{\sqrt{y^2 + r^2}} = K_0(ar) \]

Magnus et al. (1966) p. 130

\[ K_0 \left( \sqrt{a^2 + b^2 - 2ab \cos \theta} \right) = I_0(a\theta) K_0(ab) + 2 \sum_{n=1}^{\infty} \cos (n\theta) I_n(a\theta) K_n(ab) \]

where \(|a| < |b|\).


\[ e^{-e^{\cos \theta}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \cos n\theta \]

Abramowitz & Stegun (1964) p. 376 No. 9.6.35.

\[ e^{(2x + \lambda z) + i\theta \phi} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(\lambda R)^m |m|!}{(n+|m|)!} P_n^m(\cos \theta) e^{i\phi m (\psi - \psi)} \]

where

\[ R = \sqrt{x^2 + y^2 + z^2}, \quad \lambda^2 = \alpha^2 + \beta^2 \]

\[ \phi = \arctan(y/x), \quad \theta = \arccos(z/R), \quad \psi = \arctan(\beta/a) \]

Obtained by changing the argument of the exponential function into spherical polar co-ordinates and using result 6 of the table to evaluate the integrals which arise in the normal manner when a function is expanded in terms of spherical harmonics.

The method of solving the field equations used by Hohmann (1971, 1973) is based on the theory of integral equations. In this method the unknown field quantity need only be found over a small region in contrast to the half space as is required by the Finite element or Network solution methods. The method has the disadvantage that only small structures can be modelled because, for a heterogeneity in a layered medium, the solution depends on solving a set of simultaneous equations whose order is proportional to the third power of the size of the heterogeneity.

In the following sections of this paper an alternative method will be described for obtaining useful solutions to the problem of finding the potential about a point electrode in a halfspace or layered medium that has a heterogeneity embedded within it. The method is based on solving an integral equation over the surface of the heterogeneity. Thus, although the method is dependent upon solving a set of simultaneous equations, it has the advantage that the order of the system is proportional to the size of the heterogeneity squared.
Section two of this paper is devoted to deriving the integral equation. In Section three the solution of the classic problem of finding the potential of a point electrode in a two-layered ground is obtained from the integral equation derived in the preceding section. The purpose of this analysis is to provide a check on the integral equation and to provide a starting place for a brief analysis of the potential functions about two-dimensional structures. Sections four and five are devoted to the more complex problems of finding the potential about a point electrode to the surface of a layered medium that has a cylindrical or spherical conductor embedded within it.

Finally, it is shown that once the potential functions are known for the point electrode in the environment described above, it is a simple matter to obtain the apparent resistivity and IP response for a variety of electrode configurations. The likely convergence of the solution is also discussed.

2. Derivation of the integral equation for resistivity prospecting

Consider the geometry shown in Fig. 1. Let \( J \) denote the current density, \( E \) the electric field and \( \sigma \) the conductivity. From Maxwell's equations and Ohm's law:

\[
J = \sigma E
\]

and

\[
\nabla \times E = 0
\]

Therefore there exists a potential function \( U \) such that

\[
E = -\nabla U
\]

(Weatherburn 1966, p. 44).

Combining equations (1), (2) and (3) then yields:

\[
\nabla^2 U = - \left[ \nabla \cdot J + \nabla U \cdot \nabla \sigma \right] / \sigma.
\]

If \( U^* \) is the potential for the ground without the conductor, as shown in Fig. 1(a) and (c), and \( \sigma^* \) the corresponding value of the conductivity, then:

\[
\nabla^2 (U - U^*) + \nabla (U - U^*) \cdot \frac{\nabla \sigma^*}{\sigma^*} = 0
\]

outside the homogeneity,

and

\[
\nabla^2 (U - U^*) + \nabla (U - U^*) \cdot \frac{\nabla \sigma^*}{\sigma^*} = \frac{-\nabla U \cdot \nabla \sigma}{\sigma}
\]

in the homogeneity.

Let \( G \) be a solution of the following equation and subject to the same boundary conditions as \( U^* \):

\[
\nabla^2 (x - x_1) \delta(y - y_1) \delta(z - z_1).
\]

The point \((x, y, z)\) lies within the inhomogeneity. An integral equation can be found by multiplying equations (5) and (6) by \( G \) and equation (7) by \((U - U^*)\). Next subtract equation (7) from equations (5) and (6) and integrate over the region \( \Sigma > 0 \).
Whence,

\[ U = U^* + \int v G \nabla U \frac{\nabla \sigma}{\sigma} dv. \]  

(8)

Here \( v \) denotes the volume of the inhomogeneity. Following an analogous procedure to that used by Lee (1972) equation (8) is simplified to

\[ U = U^* + 2k \int s \frac{\partial u}{\partial n} Gds. \]  

(9)

In equation (9), \( s \) is the surface of the inhomogeneity, \( k = (\sigma^* - \sigma)/(\sigma^* + \sigma) \), \( \sigma \) is the conductivity of the inhomogeneity and the derivative \( \partial u/\partial n \) denotes the derivative with respect to the outward normal \( n \), of the surface \( s \). Equation (9) is the integral equation describing the electric potential for all such geometries as depicted in Fig. 1 (b) and (d). As a check on the validity of this equation, and to obtain some idea of the likely rate of convergence of the method of successive approximations for the solution, the known potential function for a two-layered ground will be found directly from equation (9) by that method.
3. Example No. 1—a two-layered ground

In this example the inhomogeneity is that region defined by $z > t$ (see Fig. 1(c)), and the required Green's function is result (5a) of Table 1. Equation (9) then becomes

$$U = U_1 - \frac{2k_1}{4\pi} \int_s \frac{\partial U}{\partial z} \left[ \frac{1}{P_1} + \frac{1}{P_2} \right] ds. \quad (10)$$

Here

$$k_1 = \frac{(\rho_2 - \rho_1)}{2(\rho_2 + \rho_1)}, \quad U_1 = \frac{I\rho_1}{2\pi R} \quad \text{(result 3 of Table 1)}$$

$$R = \sqrt{[(x^2 + y^2 + z^2)]},$$

and

$$P_1 = \frac{1}{\sqrt{[(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2]}}$$

and

$$P_2 = \frac{1}{\sqrt{[(x-x_1)^2 + (y-y_1)^2 + (z+z_1)^2]}}$$

and $s$ is the plane $z = t$. (See Fig. 1(c)).

This degenerate form of equation (9) has been known for some time (Keller & Frischknecht 1966, p. 188–194) and Pratt (1972) has considered the solution of it by a Neumann Series. That is, if $U^i$ is the $i$th approximation to the solution then the $U^{i+1}$ approximation is given by:

$$U^{i+1} = \frac{I\rho_1}{2\pi R} - \frac{k_1}{2\pi} \int_s \frac{\partial U^i}{\partial z} \left[ \frac{1}{P_1} + \frac{1}{P_2} \right] ds \quad (11)$$

and

$$U^1 = \frac{I\rho_1}{2\pi R} - \frac{k_1 I\rho_1}{(2\pi)^2} \int_s \left( \frac{\partial}{\partial z} \frac{1}{R} \right) \left[ \frac{1}{P_1} + \frac{1}{P_2} \right] ds \quad (12)$$

$$= \frac{I\rho_1}{2\pi R} + \frac{k_1 I\rho_1}{(2\pi)^2} \int_s \frac{z_1}{R^3} \left[ \frac{1}{P_1} + \frac{1}{P_2} \right] ds, \quad (13)$$

Using result 2 of Table 1 yields:

$$U^1 = \frac{I\rho_1}{2\pi R} + \frac{k_1 I\rho_1}{2\pi} \left[ \frac{1}{\sqrt{[(x^2 + y^2 + (t+|z+t|)^2)]}} \right.$$  

$$\left. + \frac{1}{\sqrt{[(x^2 + y^2 + (t-|z-t|)^2)]}} \right]. \quad (14)$$
Repeating this process \( n \) times, each time using result 2 of Table 1, yields:

\[
U^n = \frac{I\rho_1}{2\pi R} + \frac{I\rho_1}{2\pi} \sum_{m=1}^{n} \left[ \frac{k_1^m}{\sqrt{[(x^2+y^2+(2mt+z)^2)]}} + \frac{1}{\sqrt{[(x^2+y^2+(2mt-z)^2)]}} \right]
\]

where \( z < t \). This result was found by Hummell (1929, p. 232) by using the method of images.

Although the very much more complex models of spherical or cylindrical structures in a layered medium will be treated below, the above analysis does give some insight into the complicated two-dimensional problem.

Lee (1972) studied the asymptotic expansion of the Hankel transform of equation (13) for \( z = 0 \), and an arbitrary two-dimensional structure. The conclusion reached was that the transform had the same asymptotic expansion as would be obtained from equation (14). That is, the structure in the transform domain for large values of the parameter behaves as a simple planar surface at the minimum distance from the current electrode. This is because the asymptotic expansion of the Hankel transform, for large parameter, determines the potential function for small electrode separation (Griffith 1954). Hence for small values of electrode separation only the nearest parts of the structure are seen. Therefore, if equation (10) should be solved by a Neumann series the method will give more accurate results, for the potential at the top of the structure. Although Pratt (1972) observed these effects in his numerical studies, he tried to relate them to the geometry of the model rather than to the method of solving the equations.

4. Example 2 and 3—a cylindrical structure in a layered medium

Equation (9) can be readily solved for the models shown in Fig. 1(b) and (d) when the inhomogeneity is a cylindrical structure of radius \( b \). Here, the axis of the cylinder

![Fig. 2. The geometry of a cylindrical conductor in a conductive half space and a conductive two-layered medium. The point \((r_1, \theta_1)\) is the position of the singularity of the Green's function and the point \((r, \theta)\) is the observation point.](http://gji.oxfordjournals.org/)

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is taken parallel to the $x$ axis and is defined by the line $C (y = d, z = h)$. See Fig. 2. Huber (1949) was the most recent writer to attempt to solve this problem, yet he was only able to evaluate the expressions he obtained when the current electrodes were far removed from the cylinder, or when the current density near the cylinder was nearly uniform.

Besides being an important problem in its own right, the solution to this problem can also be used to check the accuracy of the more general methods of solving resistivity problems, notably the finite element approach of Coggan (1971) or the network solutions (Geoscience 1965).

For a cylinder in a half-space equation (9) becomes:

$$U = U_1 + \frac{k_1}{2\pi} \int_s \frac{\partial U}{\partial r} \left[ \frac{1}{P_1} + \frac{1}{P_2} \right] ds$$  \hspace{1cm} (16)$$

where $s$ is the surface of the cylinder, $k_1 = (\rho_2 - \rho_1)/\rho_2 + \rho_1$ and

$$U_1 = \frac{I\rho_1}{2\pi R}.$$  

Let $U$ denote

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} U e^{i\lambda x} dx.$$  

The $x$ dependence of $U$ in equation (16) can be transformed out by using the convolution theorem for Fourier integrals (Ditkin & Prudnikov 1965, p. 9) and result 7 of Table 1. Therefore since $U$ and the Green's function are symmetric in $x$, equation (16) is reduced to:

$$U_1 = \frac{1}{\pi} \int l \int \left[ K_0(|r_1|) + \frac{1}{\lambda} \int_{-\infty}^{\infty} e^{-i(\lambda z + \lambda z_1) + i\beta(\gamma - \gamma_1)} d\beta \right] dl.$$  \hspace{1cm} (17)$$

In this expression $l$ denotes the circumference of the cylinder and

$$r_1^2 = (y - y_1)^2 + (z - z_1)^2.$$

To find the unknown function $U$ across the cylinder we set

$$U = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{i\lambda x} \sum_{n = -\infty}^{\infty} e^{i\lambda n} I_n(ar) A_n dx.$$  \hspace{1cm} (18)$$

That is

$$\bar{U} = \sum_{n = -\infty}^{\infty} e^{i\lambda n} I_n(ar) A_n.$$  \hspace{1cm} (19)$$

Here $r$ and $\theta$ are cylindrical co-ordinates whose origin is at the point $(a, d, h)$ and whose axis is the line $C$ in Fig. 2.
Substituting the expression for $U$, from equation (19), into equation (17), changing the co-ordinate system to cylindrical co-ordinates, (shown in Fig. 2) and using results 8 and 9 of Table I, and the orthogonality property of the trigonometric function yield:

$$U = U_1 + k_1 \sum_{n = -\infty}^{\infty} b A_n I_n(\alpha r) \left[ 2 \frac{\partial}{\partial b} K_n(\alpha b) e^{i\theta} + \int_{-\infty}^{\infty} \frac{e^{-2\lambda h}}{\lambda} . e^{i(h - z) + i\beta y} (-1)^n \frac{\partial}{\partial b} I_n(\beta b) . e^{in\phi} d\beta \right]$$

(20)

where $\cos \psi = -\lambda/\alpha$ and $\sin \psi = i\beta/\alpha$.

A series of simultaneous equations for the $A_n$ can be found by multiplying both sides of equation (20) by $e^{-in\phi}$ and integrating around the circumference of the cylinder in a manner analogous to that described above.

Therefore

$$\frac{1}{2\pi} \int \left( \frac{2}{\pi} K_m(\alpha \sqrt{(h^2 + d^2)}) e^{-im\phi} \right) = \sum_{n = -\infty}^{\infty} b A_n \delta_{nm} \left[ 1 - 2k_1 \frac{\partial}{\partial b} K_n(\alpha b) \right] - A_n k_1 (-1)^{n+m} \frac{\partial}{\partial b} I_n(\alpha b).$$

$$= \sum_{n = -\infty}^{\infty} b A_n \delta_{nm} \left[ 1 - 2k_1 \frac{\partial}{\partial b} K_n(\alpha b) \right] - A_n k_1 (-1)^{n+m} \frac{\partial}{\partial b} I_n(\alpha b).$$

$$\times \int_{-\infty}^{\infty} \frac{e^{i\phi(m + n) - 2\lambda h}}{\lambda} d\beta \right]$$

(21)

In practice the series in equation (21) is truncated at $n = -N$ and $n = N$ to yield estimates $A_n$ for $A_n$.

Once these estimates have been found for $A_n$ they can be inserted into equations (18) and (19) to yield an expression for $U$. Hence:

$$\bar{U} = U_1 + k_1 \sum_{n = -N}^{N} b A_n \left[ K_n(\alpha r) \left( 2 \frac{\partial}{\partial b} I_n(\alpha b) \right) e^{i\theta} + \int_{-\infty}^{\infty} e^{-2\lambda h - \lambda(h - z) + i\beta y} (-1)^n \frac{\partial}{\partial b} I_n(\beta b) e^{in\phi} d\beta \right].$$

(22)

Repeating the above process for various values of $\alpha$ yields a sequence of $A_n$. From these $A_n$ the potential function can be found by first evaluating equation (22) and then equation (18).

The integral equation for the electric potential of a cylindrical conductor embedded in a half space below a conductive overburden involves the same type of integrals as the simpler case described above. Therefore a procedure exactly analogous to that described above may be used to determine the potential function.
Hence, by using the results of Table 1, the set of simultaneous equations for the unknown function $A_n$ is found by inspection of equation (21) to be given by:

$$
\frac{1}{4\pi} \int \left( \frac{2}{\pi} \right) \int_{-\infty}^{\infty} \frac{e^{-\lambda h + i\beta d}}{(1-k_1 e^{-2\lambda t})} (-1)^m e^{im\beta} d\beta
$$

$$
= \sum_{n=-\infty}^{\infty} A_n \left[ \delta_{nm} - 2k_2 \frac{\partial}{\partial b} K_n(ab) - k_2 (-1)^{n+m} \frac{\partial}{\partial b} I_n(ab) \right]
$$

$$
\times \int_{-\infty}^{\infty} \frac{e^{i\phi(m+n)2 - \lambda h}}{\lambda(1-k_1 e^{-2\lambda t})} \left[ 1 - k_1 e^{2\lambda t} \right] d\beta
$$

where

$$
k_1 = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)
$$

$$
k_2 = (\rho_3 - \rho_2)/(\rho_3 + \rho_2).
$$

See Fig. 2.

To find the expression for $U$, at the Earth's surface, it is necessary to be able to expand the Green's function $G$ (result of Table 1) in the manner described above. Since the integral form of this Green's function is of the same form as the one used previously, the expression for $U$ can be obtained from inspection of equation (20). Therefore

$$
U = U_{12} + k_2 \sum_{n=-\infty}^{\infty} b \left( A_n e^{-\lambda h + i\beta(y-d)} (-1)^n \frac{\partial}{\partial b} I_n(ab) \right)
$$

$$
\times e^{im\beta} \left[ \frac{1 - k_1}{1 - k_1 e^{-2\lambda t}} \right] \left[ e^{-2x} + e^{2x} \right] d\beta.
$$

Similarly, once the set of equations given in equation (22) has been solved a number of times for the $A_n$ for various values of $\alpha$, then the various estimates of $A_n$’s can be inserted into equation (24) to yield expressions for $U$.

5. Example 4 and 5—a spherical conductor in a layered medium

The problem of finding the potential about a point electrode on the surface of a conducting half space which contains a spherical conductor was first solved by Lipskaya (1949) by the use of bipolar co-ordinates. Since that time the resulting expressions have been evaluated for particular cases by Van Nostrand (1953), Menkel & Alexander (1971), Large (1971) and Snyder & Merkel (1973). An alternative approach has also been used by Scurtu (1972) who employed the method of images.

Dieter, Paterson & Grant (1969) also considered this problem as a particular case of the problem of determining the potential about a point electrode at the surface of a half space which contains an ellipsoidal conductor. For the simpler case of a spherical conductor the method adopted by Dieter et al (1969) worked extremely well.

The problem of finding the potential about a point electrode at the surface of a two-layered medium in which there is embedded a conducting spherical conductor does not appear to have been solved. It will be shown below that a method very
similar to that used in the two previous examples is capable of providing solutions for the potential function due to a point electrode in a half space or a two-layered medium in which a spherical conductor is embedded.

The initial geometry which will be considered is shown in Fig. 1(b). Here the material of resistivity $\rho_2$ is contained in a sphere of radius $b$. See Fig. 3(a).

For this geometry, equation (9) becomes:

$$U = U_1 + k_1 \int_{\text{sphere}} \frac{\partial U}{\partial b} \left[ \frac{1}{P_1} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x+\tau_1) + i(\tau-x_1)} \times \frac{e^{iy_1\rho}}{\lambda} \, d\sigma d\phi \right].$$

This integral equation is solved by solving for the coefficients $A_n^m$ in the expression for $U$ given below.

$$U = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} R_1^n A_n^m e^{im\phi} P_n^{|m|} (\cos \theta_1).$$

This expression for $U$ is the one that would be obtained if Laplace's equation were solved for $U$ within the spherical conductor. The symbols are defined in Fig. 3.

Substituting the expression for $U$ given in equation (26) into equation (25), changing from the rectangular co-ordinate system to spherical co-ordinates system.

Fig. 3. The geometry of a spherical conductor in a conductive half space and a conductive two-layered medium. The point $(R_1, \theta_1, \phi_1)$ is the position of the singularity of the Green's function and the point $(R, \theta, \phi)$ is the observation point. $R_0 = \sqrt{d^2 + h^2}$, $\phi_0 = \pi$ and $\theta_0 = \arctan(d/h)$. 
(shown in Fig. 3), and using result 10 of Table 1 and the orthogonality of the Legendre functions yield:

\[
U = U_1 + 2k_1 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{nb^{n+1} A_n^m}{2n+1} \left[ \frac{1}{b} \left( \frac{R}{b} \right)^n P_n^{|m|} (\cos \theta) e^{im\phi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda x - \lambda h - iax - ibx - ibd} (b\lambda)^n i^{|m|} e^{im\phi} v(n+|m|)! (-1)^{n-m} d\alpha d\beta \right].
\]

(27)

A set of simultaneous equations for the unknown parameters \( A_n^m \) may now be found from equation (27) by multiplying both sides of equation (27) by \( P_n^{|m|}(\cos \theta) \sin \phi e^{-im\phi} \) and integrating over the sphere. The integrations are readily performed by using result 10 of Table 1 and the orthogonality property of the spherical harmonics. Therefore for \( \mu \) and \( \nu \) arbitrary

\[
\frac{I\rho_1}{2\pi} \left( \frac{1}{R_0} \right)^{\mu+1} P_\mu^\nu (\cos \theta_0) (u+|v|)! e^{-iv\phi_0} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_n^m \left[ \delta_{nm} \delta_{\mu\nu} \frac{2k_1 n}{2n+1} - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2k_1 e^{-\lambda x + \lambda h - iax - ibx - ibd} \lambda^\mu (b\lambda)^n nb^{n+1} i^{|m|} i^{|m|}}{\lambda(2n+1) (n+|m|)! (n+|v|)!} (-1)^{n+|m|} d\alpha d\beta \right].
\]

(28)

Once estimates, \( \tilde{A}_n^m \), for the coefficients have been found from equation (28) by truncating the series after \( N \) terms, they are inserted into equation (27) to yield:

\[
U = U_1 + 2k_1 \sum_{n=0}^{N} \sum_{m=-n}^{n} \frac{n(b)^n}{2n+1} \tilde{A}_n^m \left[ \frac{1}{R} \left( \frac{b}{R} \right)^n P_n^{|m|} (\cos \theta) e^{im\phi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\lambda x - \lambda h - iax - ibx - ibd} \lambda^\mu (b\lambda)^n nb^{n+1} i^{|m|} i^{|m|}}{\lambda(n-|m|)! (n+|v|)!} (-1)^{|m|} d\alpha d\beta \right].
\]

(29)

Equation (28) is then the required solution of equation (25).

The integral equation for the electric potential of a spherical conductor embedded in a half space below a conductive overburden involves the same type of integrals as the simpler case described above. Therefore a procedure exactly analogous to that described above may be used to determine the potential function.

Hence by using the results of Table 1 the set of simultaneous equations for the unknown functions \( A_n^m \) is found by inspection of equation (28) to be given by:

\[
\frac{\lambda^\mu I\rho_1}{2\pi} \cdot \frac{i^{|v|}(-1)^{|v|}}{(n+|v|)!} \cdot \frac{(1+k_1)}{2\pi} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\lambda x + \lambda h - iax - ibx - ibd} \lambda^\mu (b\lambda)^n nb^{n+1} i^{|m|} i^{|m|}}{\lambda(1-k_1 e^{-2\lambda\nu})} d\alpha d\beta
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_n^m \left[ \delta_{nm} \delta_{\mu\nu} \frac{2k_2 n}{2n+1} - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2k_2 e^{-iv\phi + im\phi - 2\lambda h} \lambda^\mu i^{|v|} i^{|m|}}{(1-k_1 e^{-2\lambda\nu})(2n+1) (n-|m|)! (n+|v|)!} \right]
\times \times (-1)^{|m|+n} b^{n} d\alpha d\beta
\]

(30)
where \[ k_1 = \frac{(\rho_2 - \rho_1)}{\left(\rho_2 + \rho_1\right)} \]
and \[ k_2 = \frac{(\rho_3 - \rho_2)}{\left(\rho_3 + \rho_2\right)} \]

See Fig. 3.

To find the expression for \( U \) outside the sphere, it is necessary to be able to expand the Green's function \( G \) (result 5(b) of Table 1) in the manner described above. Since the integral form of this Green's Function is of the same form as the one used previously, the expression for \( U \) can be written down from inspection of equation (29). Therefore

\[
U = U_{12} + 2k_1 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{nb^{n+1}}{2n+1} A_n^m \left[ \frac{1}{R} \left( \frac{b}{R} \right)^n P_n^m (\cos \theta) e^{i m \phi} + \frac{(1-k_1)}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-2k \lambda} \lambda (n-m)!}{(1-k_1 e^{-2k \lambda})} \lambda (n-m)! \right] (31)
\]

In practice the infinite set of simultaneous equations described by equation (30) is truncated after a finite number of terms and the resulting estimates \( A_n^m \) are inserted into equation (31) in order to yield an estimate for the potential function \( U \).

6. Discussion

Once the potential function, \( U \), is known for a point electrode in any of the environments described previously, then the apparent resistivity, \( \rho_a \), can be found as follows. If \( U \) is measured, at the points \((x_1, y_1, 0)\) and \((x_2, y_2, 0)\) on the earth's surface, then, as Keller & Frischknecht (1966) have shown,

\[
\rho_a = \left( U(x_1, y_1, 0) - U(x_2, y_2, 0) \right) K/I. \quad (32)
\]

Here \( K \) is a geometric factor that is determined by the electrode arrangement.

According to Siegel (1959), if \( M_a \) is the apparent chargeability and \( M_i (i = 1, \ldots, n) \) are the chargeabilities of the various media, then \( M_a \) can be found from:

\[
M_a \approx \sum_{i=1}^{n} M_i \frac{\partial \log \rho_a}{\partial \log \rho_i} \quad (33)
\]

The identity

\[
\sum_{i=1}^{n} \frac{\partial \log \rho_a}{\partial \log \rho_i} = 1
\]

can be used to simplify equation (33). Hence, once the potential about the point electrode is known, then the apparent resistivity and chargeability can be readily calculated from equations (32) and (33), respectively.

All the above discussion has assumed that the number of unknown functions \( A_n \) and \( A_n^m \) of equations (19) and (26) respectively is known. There does not, however,
appear to be any general procedure for determining the number of required modes despite the fact that the process advocated above is known to converge correctly (Ikebe 1972). The usual approach, then, that has been employed by Howard (1972) and Lee (1974), who have used the Galerkin method to solve the more difficult integral equations of electromagnetic and transient electromagnetic problems, is to increase the number of modes until convergence is achieved. This procedure was found to work very well and the numbers of modes required for those problems was about 8.

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References


