Anisotropic resistivity tomography

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SUMMARY
Geophysical tomographic techniques have the potential to remotely detect and characterize geological features, such as fractures and spatially varying lithologies, by their response to signals passed through these features. Anisotropic behaviour in many geological materials necessitates the generalization of tomographic methods to include anisotropic material properties in order to attain high-quality images of the subsurface. In this paper, we present a finite element (FE) based direct-current electrical inversion method to reconstruct the conductivity tensor at each node point of a FE mesh from electrical resistance measurements. The inverse problem is formulated as a functional optimization and the non-uniqueness of the electrical inverse problem is overcome by adding penalty terms for structure and anisotropy. We use a modified Levenberg–Marquardt method for the functional optimization and the resulting set of linear equation is solved using pre-conditioned conjugate gradients. The method is tested using both synthetic and field experiments in cross-well geometry. The acquisition geometry for both experiments uses a cross-well experiment at a hard-rock test site in Cornwall, southwest England. Two wells, spaced at 25.7 m, were equipped with electrodes at a 1 m spacing at depths from 21–108 m and data were gathered in pole–pole geometry. The test synthetic model consists of a strongly anisotropic and conductive body underlain by an isotropic resistive formation. Beneath the resistive formation, the model comprises a moderately anisotropic and moderately conductive half-space, intersected by an isotropic conductive layer. This model geometry was derived from the interpretation of a seismic tomogram and available geological logs and the conductivity values are based on observed conductivities. We use the test model to confirm the ability of the inversion scheme to recover the (known) true model. We find that all key features of the model are recovered. However, the inversion model is smoother than the true model and the difference in absolute value of anisotropy and conductivity between features is slightly underestimated. Using an anisotropic conductivity distribution aggravates the problem of non-uniqueness of the solution of the inverse electrical problem. This problem can be overcome by applying appropriate structural and anisotropy constraints. We find that running a suite of inversions with varying constraint levels and subsequent examination of the results (including the inspection of residual maps) offers a viable method for choosing appropriate numerical values for the imposed constraints. Inversion of field data reveals a strongly anisotropic subsurface with marked spatial variations of both magnitude of anisotropy and conductivity. Average conductivities range from 0.001 S m⁻¹ (≈ 1000 Ω m) to 0.003 S m⁻¹ (≈ 333 Ω m) and anisotropy values range from 0 per cent to more than 300 per cent. As an independent test of the reliability of the structures revealed by anisotropic electric tomography, anisotropic seismic traveltime tomograms were calculated. We find a convincing structural agreement between the two independently derived images. Areas of high electric anisotropy coincide with seismically anisotropic areas and we observe an anticorrelation between electric conductivity and seismic velocity. Both observations are consistent with anisotropy anomalies caused by fracturing or layering.

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1 INTRODUCTION

Anisotropy of the physical parameters of rocks can be caused by fracturing, layering, stress and the alignment of crystals, grains and pore space. Fig. 1(a) shows a strongly layered rock sample from a hydrological test site in Cornwall, southwest England. Electrical anisotropy measured on this sample is marked with minimum and maximum resistivities observed ranging from 330 Ω m to 1240 Ω m. Furthermore anisotropy can also be spatially varying in magnitude and direction. Spatially varying anisotropy is clearly illustrated in Fig. 1(b), where fracture sets create an anisotropic subsurface. In the foreground the fracture set is oriented diagonally, reorienting itself towards the back, eventually being aligned parallel with the sides of the image. Besides a change in orientation of the fracture, the fracture density is spatially varying, with a low fracture density in the left part of the image and increasing fracture density towards the right. Thus anisotropy at this site is heterogeneous in both direction and amplitude.

When reconstructing subsurface models from electrical data acquired at the Earth’s surface, it can be difficult to separate the effects of anisotropy from effects caused by varying layer thickness (principle of equivalence, see Maillet (1947)) and it may be difficult to resolve anisotropy. In this context it is important to note that the choice of acquisition geometry has a major impact on the ability to resolve anisotropy in inversion images. For example in reflection seismology, the importance of including anisotropy in seismic data processing and the ability to resolve anisotropy parameters by seismic data processing and imaging was realized only when wide-angle (large-aperture) data sets were used (Tsvankin & Thomsen 1995).

Cross-well data are generally acquired with a large aperture (i.e. data are acquired over a wide range of view angles) and are therefore well suited for observing anisotropic physical properties. In seismic cross-well imaging this has been realized for more than 10 yr and a number techniques to reconstruct the anisotropic seismic velocity distribution between boreholes have been developed (Stewart 1988; Chapman & Pratt 1992; Pratt & Chapman 1992; Michelena et al. 1993, 1995; Abraham et al. 1998).

In electrical prospecting, anisotropy has also been known to exist for a long time (Maillet 1947) and the influence of anisotropy on electrical observations is well-documented in the literature. Resistivity anisotropy has been observed in field studies for applications in both groundwater flow estimation (Ritzi & Andolsek 1992) and well-logging (Schön et al. 2000). The observed anisotropy is usually ascribed to fracturing or microlayering. Subsequently, fracture, pore space and layering models have been developed to predict anisotropy parameters for a given subsurface model (e.g. Campbell 1977; Bahr 1997; Schön et al. 2000).

Efforts have been made to incorporate anisotropy in electrical inversion algorithms. For example, Jupp & Vozoff (1977) overcome the non-uniqueness caused by anisotropy by combining electrical and electromagnetic data, where the electrical data are sensitive to a combination of the vertical and the horizontal component of the conductivity tensor and the electromagnetic data are only sensitive to the horizontal component of the conductivity tensor. Since electric cross-well data are acquired over a large range of view angles, these data are sensitive to both vertical and horizontal components of conductivity, and it is feasible to reconstruct anisotropic electrical properties from direct current (DC) electrical measurements alone.

In 1998, a series of electric and seismic cross-well experiments at the Reskajeage Quarry hydrological test site (Cornwall, southwest England) were conducted. The resulting electrical resistance and seismic traveltime data showed strong anomalies from both (1) laterally and vertically varying structures and (2) anisotropic physical properties (Herwanger 2001). Anisotropic seismic tomosgrams displayed strong anisotropy of up to 35 per cent. The electrical anisotropy led to spurious artefacts in isotropic electrical tomosgrams in the form of unreasonably banded images. The presence of strong anisotropy at the field site, probably caused by both fracturing and layering, and the severity of artefacts caused by neglecting anisotropy have led us to develop a tomographic inversion algorithm in order to reconstruct the electrical conductivity tensor in two or three dimensions.

In this paper we briefly describe the methodology of anisotropic resistivity tomography and then present synthetic data and field data
inversion results. In Section 2 we describe the methods employed for solving the forward and inverse problem, with special emphasis on regularization terms (Section 2.3). The experiment that triggered the development of the presented algorithm is described in Section 3. We use two strategies to assess the performance characteristics and predictive power of the newly developed inversion algorithm for field data sets. First, we use computer-generated data calculated using the same source–receiver geometry and using a subsurface model similar to the expected subsurface at the Reska-jeage Quarry field site. Using computer-generated data, the true subsurface model is known and can be compared with inversion models. Secondly, we conducted a seismic tomography experiment, scanning the same region as the electrical experiment. This allows us to compare the subsurface structure derived from anisotropic electrical inversion with the structure derived from anisotropic seismic inversion. The synthetic data inversion tests are presented in Section 4. In a first test (Section 4.1) we show how the inclusion of a priori assumptions of model covariance influences inversion models. Then we present possible improvements to inversion by fine-tuning inversion parameters. Subsequently we present the same suite of inversion tests using field data (Section 5) and show that field data inversion and synthetic data inversion exhibit essentially the same performance characteristics. To independently assess the quality of the electrical inversion image, we compare tomographic inversion images from (anisotropic) seismic traveltime tomography with the images derived using the new algorithm (Section 5.2). In a final section (Section 6), we discuss the results and give recommendations of possible applications of the novel inversion technology.

2 METHOD

In this section, we describe the inversion strategy to reconstruct a conductivity tensor function from DC electrical experiments. The method is described in detail in Pain et al. (2003). Here we limit ourselves to theory necessary for the understanding of the remainder of the paper, with a special emphasis on the description of anisotropic conductivity (Section 2.2) and practical ways to include a priori information in the error functional (Section 2.3).

2.1 Forward problem

In geoelectrical experiments current is driven into the ground between a pair of current electrodes and the resulting potential field is monitored by measuring the voltage between a pair of potential electrodes (Fig. 2). Mathematically the potential field $U$ generated by a current of strength $I$ injected between electrodes at locations $r_{s1}$ and $r_{s2}$ into a medium with conductivity distribution $\sigma(r)$ is described by

$$U(r) = \frac{I}{4\pi \sigma(r)} \ln \left( \frac{r_{s2}}{r_{s1}} \right).$$

![Figure 2](https://example.com/fig2.png)

**Figure 2.** Sketch of survey geometry for electrical cross-well experiment in pole–pole geometry. Current is injected in the right borehole (BH19) and the resulting potential field is schematically displayed by the stippled lines. The displayed survey geometry is used for both synthetic and field data inversions. The location of the boreholes is drawn to scale. A total of 88 current electrodes are deployed in the right borehole BH19 and the resulting potentials are monitored at 87 electrodes in BH20. The region between the boreholes marked by the dark shade of grey is used to display the tomograms in Figs 3, 5 and 8.
by the Laplace equation:
\[
\nabla \cdot \sigma(r) \nabla U(r) = I\delta(r - r_{p1}) - I\delta(r - r_{p2}).
\]

Note that the conductivity distribution \(\sigma(r)\) is both inhomogeneous (i.e. a function of the location \(r\) in the medium) and anisotropic (i.e. at a fixed location dependent on the direction of observation). The voltage \(V\) observed between two potential electrodes located at \(r_{p1}\) and \(r_{p2}\) is calculated as the difference in potential \(V = U(r_{p1}) - U(r_{p2})\).

Analytical solutions to eq. (1) exist only for a limited number of simple geometries of conductivity models. One way to solve eq. (1) in arbitrarily inhomogeneous media is discretization by finite elements (FE). The following is a brief summary of the techniques employed. For a comprehensive treatment of the FE method and concepts see, for example, Zienkiewicz & Taylor (2000).

We use a node-wise FE description whereby the values of the conductivity tensor \(\sigma\), the current \(I\) and the resulting potential field \(U\) are sampled at the \(N\) node points of the FE mesh. In the region between node points values are linearly interpolated (in three dimensions) by trilinear basis functions \(Q_i\), with a total number of \(N\) basis functions, each associated with one node. The potentials \(U\) observed at the node points can be arranged into a vector \(u\). We use isoparametric hexahedral elements, i.e. every element is bounded by six planar faces, and there are eight nodes associated with each element. Using the Galerkin method, that is using weighting functions, that is they are non-zero only in the elements containing \(u\) is sparse with 27 non-zero elements per row. The integration in eq. (3) needed to calculate each matrix element \(A_{ij}\) is solved numerically by Gaussian quadrature.

\[
Au = b
\]

with the elements \(A_{ij}\) of matrix \(A\) given by
\[
A_{ij} = -\int_{V} \nabla Q_i \sigma \nabla Q_j \, d^3r
\]

\[\text{(3)}\]

and \(b\) being a suitable FE discretization of point current sources. We need to be able to locate sources at an arbitrary location in the domain (i.e. sources are not restricted to node points). We have chosen to interpolate sources via a Gaussian to the nearest node points (see Pain et al. 2002). The domain of integration in eq. (3) is the entire domain \(V\). However, since the functions \(Q_i\) and \(Q_j\) are local shape functions, that is they are non-zero only in the elements containing a specific node as a corner point, the matrix \(A\) is sparse with 27 non-zero elements per row. The integration in eq. (3) needed to calculate each matrix element \(A_{ij}\) is solved numerically by Gaussian quadrature.

\[\text{2.1.1 Boundary conditions}\]

Boundary conditions need to be imposed on the entire outer surface of the solution domain. At the Earth's surface \(\Gamma_{surf}\) the boundary condition is physically motivated. Since air is an almost perfect isolator, the current across the Earth's surface is negligible and can be equated to zero. Mathematically, current across a surface is described by the normal derivative of the potential field and the resulting Neumann boundary condition at the Earth's surface is:
\[
\frac{\partial U}{\partial n} \bigg|_{\Gamma_{surf}} = 0.
\]

The remaining boundaries of the computational domain \(\Gamma - \Gamma_{surf}\) need to be chosen somewhat arbitrarily, due to the fact that the computational domain (i.e. the extent of the FE mesh) necessarily has a finite size. In physical experiments the boundary conditions are zero potential at infinity. In our computer implementation of electrical experiments the infinite size of the domain is simulated by using exponentially growing elements near the boundary (Herwanger 2001). The Dirichlet boundary conditions of zero potential are given by:
\[
U|_{r = r_{inf}} = 0.
\]

Two alternative ways of imposing boundary conditions at the boundaries of the modelling domain are possible. Zhang et al. (1995) and Weller et al. (1996) show that mixed Robin boundary conditions allow for a smaller modelling domain while retaining the accuracy of the solution and Bing & Greenhalgh (2001) have successfully applied Robin boundary conditions using FE techniques. Alternatively, infinite elements (Zienkiewicz & Morgan 1983) could be used at the boundary. This approach models the actual physical boundary condition of zero potential at infinity. Applying these techniques would improve the computational efficiency of the proposed method, but would not alter the substance of this paper.

The Neumann boundary conditions are ‘natural boundary conditions’ (Zienkiewicz & Taylor 2000), meaning they are automatically satisfied in the formulation of the FE method we employ. The Dirichlet boundary conditions are ‘forced’ boundary conditions and are implemented with the ‘big spring’ method (Cheung et al. 1996).

Once all elements of \(A\) (including the boundary conditions) and \(b\) are assembled, eq. (2) can be solved for \(u\). For this task we use a pre-conditioned conjugate gradient (PCG) solver (Golub & Van Loan 1996). Employing a PCG solver has the advantage of being memory efficient on sparse matrix equations and systems containing several hundred thousand unknowns can be readily solved on today’s computers. Once the solution for the potentials at each node point \(u\) has been obtained, these values can be interpolated to the values of the potentials at the receiver locations (which are not necessarily located at a node point).

Note that eq. (2) has to be solved for each source vector \(b\). Using direct solvers (i.e. solvers in which the inverse matrix of \(A\) is formed explicitly), would have the advantage that once an inverse matrix \(A^{-1}\) is found, the solution for the potentials \(u\) with a new source vector \(b\) reduces to a (fast) matrix-vector multiplication. However, the inverse matrix \(A^{-1}\) is not sparse and memory requirements become prohibitive.

\[\text{2.2 Material properties}\]

The material property describing the subsurface for DC electrical experiments is the conductivity tensor \(\sigma(r)\) (see eq. 1). In this section we describe the model vector we want to reconstruct by inversion and its relationship with the conductivity tensor.

The conductivity tensor \(\sigma\) can be decomposed into a product of a diagonal matrix containing its eigenvalues \(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3\) and rotation matrices consisting of trigonometric functions of the Euler angles \(\alpha, \beta, \gamma\) (see, for example, Arfken & Weber 1995, p. 198). The Euler angles \(\alpha, \beta, \gamma\) measure rotations around the z-axis, the \(y\)-axis and the rotated \(z\)-axis respectively.

The conductivity tensor in terms of eigenvalues and rotations can then be written as:
\[
\sigma = R^T \tilde{\sigma} R.
\]

In this expression \(R\) is the rotation matrix given by:
\[
R(\alpha, \beta, \gamma) = \begin{pmatrix}
\hat{r}_{11} & \hat{r}_{12} & \hat{r}_{13} \\
\hat{r}_{21} & \hat{r}_{22} & \hat{r}_{23} \\
\hat{r}_{31} & \hat{r}_{32} & \hat{r}_{33}
\end{pmatrix}
\]

\[\text{(7)}\]
and the model vector \( \mathbf{m} \) can be defined as:

\[
\mathbf{m} = \begin{pmatrix}
    m_1^\alpha \\
    m_2^\beta \\
    m_3^\gamma \\
    m_4^{\log \hat{\sigma}_1} \\
    m_5^{\log \hat{\sigma}_2} \\
    m_6^{\log \hat{\sigma}_3}
\end{pmatrix}
\]

Reconstructing the distribution of six material properties creates a difficulty for displaying inversion images. Displaying average conductivity and anisotropy is one way of displaying key information in an accessible form. For this purpose we define average conductivity by the arithmetic mean of the three conductivity eigenvalues

\[
\langle \sigma \rangle = \frac{\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3}{3}
\]

(14)

and percentage anisotropy

\[
\text{anisotropy} = \left( \frac{\hat{\sigma}_1}{\langle \sigma \rangle} \right)^2 \times 100 \ \text{per cent.}
\]

(15)

### 2.3 Designing the error functional

Most geophysical inverse problems are either mixed determined or underdetermined. For example in the applications in Sections 4 and 5 roughly 8000 data points are inverted to obtain a model described by about 257 000 (= 6 \times \text{the number of nodes}) model parameters. In other words, we are solving a system of equations with fewer data than model parameters. It is therefore possible to find more than one model vector \( \mathbf{m} \) that predicts the observed data. The task of a practical inversion scheme is to single out one of the many possible solutions. This one solution must use the information about the model contained in the data. At the same time, additional \textit{a priori} information must be added to limit the number of possible models to one unique inversion model. The nature of the \textit{a priori} information (or model covariance information) will determine the character of the inversion model and the inclusion of \textit{a priori} information regularizes the inverse problem (Tikhonov & Arsenin 1977; Tikhonov et al. 1995). Tikhonov regularization by means of smoothness constraints has been made popular in geophysical applications by Constable et al. (1987).

We formulate the inverse problem as a functional optimization problem by minimizing an error (or misfit) functional. The employed error functional contains two parts \( F_d \) and \( F_r \), whereby \( F_d \) ensures that the observed data is predicted by the inversion model and \( F_r \) is a regularization term, constraining the solution of the inverse problem to belong to a desired class of models. The error functional is thus written as:

\[
F = F_d + F_r,
\]

(16)

where \( F_d \) is a measure of data misfit, defined in eq. (20). The regularization functional \( F_r \) is a sum of three parts:

\[
F_r = F_{r}^d + F_{r}^a + F_{r}^l
\]

(17)

containing model covariance measures for the desired structure, anisotropy and deviation from a starting (or reference) model, respectively. Definitions of these functionals are given in Sections 2.3.2, 2.3.3 and 2.3.4.

### 2.3.1 Data misfit

Observed quantities in geoelectrical experiments are the strength \( I \) of current injected between two current electrodes and the resulting voltage \( V \) between two potential electrodes. From \( I \) and \( V \) either transfer resistances

\[
R = \frac{U}{T}
\]

(18)

or apparent resistivities

\[
\rho_a = \frac{4\pi}{G} \frac{1}{T} \frac{U}{G}
\]

(19)

can be calculated. In the latter equation \( G \) is a geometry factor (see, for example, Telford et al. (1990) for surface geometries and Herwanger (2001) for cross-well geometries). In surface electrical experiments apparent resistivities are routinely used as data for
display purposes and as input to computerized inversion. In cross-well experiments serious artefacts can result when using apparent resistivities as input data for electrical cross-well tomography (Heweranger 2001; Pain et al. 2002). Using transfer resistances as observed data (and using appropriate weights in the error functional) provides an alternative to the use of apparent resistivities. This is the approach taken here.

Using the solution vector \( \mathbf{u} \) of eq. (2), data \( d^\text{obs} \) can be predicted for the same source–receiver geometries as observed in a field experiment. The sum of the squared differences between observed and predicted data

\[
F_d = \frac{1}{2} \sum_{i=1}^{N_{\text{DATA}}} w_i \left[ d^\text{obs} - d^\text{pre}(\mathbf{m}) \right]^2
\]

(20)
is a measure to what degree a model \( \mathbf{m} \) can predict the observed data. If \( F_d \) is small, \( \mathbf{m} \) predicts the observed data well and vice versa. Each data residual \( d^\text{obs} - d^\text{pre}(\mathbf{m}) \) is scaled by a weight \( w_i \). The weights \( w_i \) can contain (if available) information about the quality of datum \( d^\text{obs} \). Typically weights of \( 1/\sigma_i^2 \) (where \( \sigma_i^2 \) is the standard deviation as measure of data error) are used. Additionally, the weights \( w_i \) can be used to balance large and small data values. If the geometry factor is used for weighting this affects the use of apparent resistivities as input data for inversion (see Heweranger 2001; Pain et al. 2002).

In matrix notation, eq. (20) can be written as:

\[
F_d = \frac{1}{2} (\mathbf{d}^\text{obs} - \mathbf{d}^\text{pre})^T \mathbf{W}_d (\mathbf{d}^\text{obs} - \mathbf{d}^\text{pre}).
\]

(21)

In this notation, the observed data and predicted data have been assembled into the vectors \( \mathbf{d}^\text{obs} \) and \( \mathbf{d}^\text{pre} \) respectively and the data weights \( w_i \) form the diagonal of the weight matrix \( \mathbf{W}_d \) and \( \mathbf{W}_d \) is the inverse of the data covariance matrix.

### 2.3.2 Structural or smoothness constraints

Electrical data measure the integrated effect of the medium that the electrical current samples and the voltages observed at a distance from a current source are insensitive to the fine-grained properties of the medium. This can be used as a justification for the use of ‘smooth’ models in the regularization of the inverse problem.

Geological materials are often layered, resulting in a smooth medium within the layers but a rough medium across layer boundaries. Using predominantly rough and smooth directions in the model space can be incorporated into smoothness constraints, if the preferential strike and dip directions of the layering are known.

In order to impose structural constraints that measure smoothness in a directionally dependent way, we have designed the following functional (see also Heweranger 2001; Pain et al. 2002):

\[
F_s = \frac{1}{2} \sum_{\mu=1}^{6} \lambda_\mu \int_V \nabla^T \mathbf{m}^\mu \mathbf{S} \nabla \mathbf{m}^\mu d^3 r.
\]

(22)

For each of the six material properties \( m^1, \ldots, m^6 \) a scalar product \( \nabla^T \mathbf{m}^\mu \mathbf{S} \nabla \mathbf{m}^\mu \) of the gradient:

\[
\nabla \mathbf{m}^\mu = \begin{pmatrix} \frac{\partial \mathbf{m}^\mu}{\partial x} \\ \frac{\partial \mathbf{m}^\mu}{\partial y} \\ \frac{\partial \mathbf{m}^\mu}{\partial z} \end{pmatrix}
\]

(23)
is calculated and integrated over the entire domain \( V \). The resulting six numbers (\( \mu = 1, 2, 3 \) for the structure in the three rotation angles and \( \mu = 4, 5, 6 \) for the structure in the three conductivity eigenvalues) are measures for the amount of ‘structure’ contained in the model. Each measure of structure is weighted by a penalty parameter \( \lambda_\mu \) and the six measures of structure are summed to form the structural penalty \( F_s^* \).

In the simplest case, \( \mathbf{S} \) is the identity matrix and the scalar product for the \( \mu \)-th material

\[
\nabla^T \mathbf{m}^\mu \nabla \mathbf{m}^\mu
\]

(24)

reduces to the squared gradient

\[
\left( \frac{\partial \mathbf{m}^\mu}{\partial x} \right)^2 + \left( \frac{\partial \mathbf{m}^\mu}{\partial y} \right)^2 + \left( \frac{\partial \mathbf{m}^\mu}{\partial z} \right)^2.
\]

(25)

In this form, the structural constraints are simply smoothness constraints, advocated for example in Constable et al. (1987).

At our field site the acquisition geometry is 2-D. We choose the plane in which the measurements are taken to be the \( x-z \) plane. Using the tensor \( \mathbf{s} \)

\[
\mathbf{s} = \begin{pmatrix} 1 & 0 & 10^9 \\ 0 & 1 & 0 \\ 10^9 & 0 & 0 \end{pmatrix}
\]

(26)

forces a homogeneous model in the \( y \) direction, i.e. for fixed \( x-z \) coordinates, the conductivity tensor does not change in the \( y \) direction. We effectively use the capability of anisotropic smoothness constraints to force a 2-D subsurface model calculated with a 3-D inversion algorithm. A graphical demonstration of the efficacy of this approach is given in Heweranger (2001) and Pain et al. (2002).

Applying a FE approximation to the functional for the structural penalty in eq. (22), the functional can be abbreviated in matrix notation:

\[
F_s^* = \frac{1}{2} \mathbf{m}^T \mathbf{S} \mathbf{m}.
\]

(27)

The structure of matrix \( \mathbf{S} \) is explained in detail in Pain et al. (2003).

### 2.3.3 Anisotropy constraints

DC electrical inversion problems are notoriously ill-posed. Introducing anisotropy necessitates the introduction of even more model parameters (in our case a factor of 6) and thus aggravates the problem of ill-posedness. Therefore it seems reasonable to only allow as much anisotropy as is necessary to fit the data in any inversion solution in order to regularize the solution. A functional that allows us to constrain solutions to create only the necessary levels of anisotropy is given by:

\[
F_a^* = \frac{1}{2} \lambda^a \int (m^4 m^5 m^6) \mathbf{a} \left( \begin{array}{c} m^4 \\ m^5 \\ m^6 \end{array} \right) d^3 r.
\]

(28)

The model parameters \( m^4, m^5 \) and \( m^6 \) contain the logarithm of the eigenvalues of the conductivity tensor as defined in eq. (13). The tensor \( \mathbf{a} \) has the form of a discretized Laplacian and one typical form is therefore:

\[
\mathbf{a} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.
\]

(29)

For the work in this paper we assume a transversely isotropic (TI) medium, such as a finely layered medium where conductivity within
the plane of layering is rotationally invariant. Here we force a TI medium by using
\[
a = \begin{pmatrix}
10001 & -10000 & -1 \\
-10000 & 10001 & -1 \\
-1 & -1 & 2
\end{pmatrix}.
\] (30)

This affects \( m^3 \) to equal \( m^5 \) (i.e. \( \hat{\sigma}_1 = \hat{\sigma}_2 \)) in order to keep the anisotropy penalty small, with the result of creating an isotropic medium in the \( m^2 \)–\( m^7 \) plane. The choice of numerical values for the components \( a_{ij} \) of \( a \) in eq. (30) is somewhat arbitrary. However, the numerical value \( a_{00} \) is not crucial, as long as the elements \( a_{11}, a_{12}, a_{21} \) and \( a_{22} \), measuring the difference in conductivity between \( m^4 \) and \( m^6 \), are very much larger than the remaining elements and the row (and column) sums of \( a \) are equal to zero. Note that the angle of the symmetry axis (with respect to the coordinate axes) of the TI medium can be spatially varying.

Using a FE approximation for eq. (28) the anisotropy penalty functional can be written in matrix form:
\[
F_a = \frac{1}{2} m^T A m.
\] (31)

For a description of the structure of matrix \( A \) see Pain et al. (2003).

2.3.4 Steplength damping

Deviation from a reference model is another possible form of regularizing an ill-posed inverse problem. A simple form of a reference model can be a homogeneous half-space model. More complex reference models can often be created by translating a geological model into a model of physical parameters, or in the case of cross-well data from interpolation of well-log data.

In an iterative inversion algorithm, either a fixed reference model or the model from a previous iteration can be used as a reference model. Here, we use the model from a previous inversion step \( m_{\text{old}} \) as a reference model. Then the deviation of an updated model \( m \) from a given a reference model \( m_{\text{old}} \) can be quantified by the following functional:
\[
F^t_i = \frac{1}{2} \sum_{\mu=1}^{6} \lambda_{\mu}^t \int_V (m^\mu - m_{\text{old}}^\mu)^2 \, d^3 r.
\] (32)

In this equation the square of the deviation of each of the six components \( m^1, \ldots, m^6 \) of the material property functions (see eq. 10) from a reference model \( m_{\text{old}}^\mu, \ldots, m_{\text{old}}^\mu \) is integrated over the domain \( V \). The factors \( \lambda_{\mu}^t \) are penalty levels. For large levels \( \lambda_{\mu}^t \), the model updates are small, and for small \( \lambda_{\mu}^t \), large model updates can result. The term \( F^t_i \) effectively limits the stepsize of a model update in an inversion algorithm resulting from minimization of \( F = F_d + F_i + F^t_i + F^f_i \), Therefore the term \( F^t_i \) is often referred to as a steplength damping term.

Applying a FE approximation, eq. (12) can be written in matrix form:
\[
F^t_i = \frac{1}{2} (m - m_{\text{old}})^T \, M (m - m_{\text{old}}).
\] (33)

Using the matrix expressions for the four functionals gauging data error (eq. 21), structural constraints (eq. 27), anisotropy constraints (eq. 31) and the deviation from a starting model (eq. 33) respectively, the error functional \( F \) (see eq. 16) can be written as:
\[
F = \frac{1}{2} (d_{\text{obs}} - d_{\text{pre}})^T W_d (d_{\text{obs}} - d_{\text{pre}}) + \frac{1}{2} m^T S m + \frac{1}{2} m^T A m + \frac{1}{2} (m - m_{\text{old}})^T M (m - m_{\text{old}}).
\] (34)

2.4 Solution of the inverse problem

We solve the inverse problem of deriving a model \( m \) from given data \( d_{\text{obs}} \) by finding a minimum of the error-functional in eq. (34). To this end, we first approximate the data error vector \( \Delta d(m) = d_{\text{obs}} - d_{\text{pre}}(m) \) by a first-order Taylor series around an initial model \( m_{\text{old}} \) and rewrite the data error of a new model \( m_{\text{new}} \) using this approximation
\[
\Delta d(m_{\text{new}}) \approx \Delta d(m_{\text{old}}) + J (m_{\text{new}} - m_{\text{old}}).
\] (35)

Hereby we make use of the Jacobian matrix \( J \) with elements defined by
\[
J_{ij} = \frac{\partial (\Delta d_i)}{\partial m_j}.
\] (36)

Using the linearized expression for the data error, we can minimize the error functional (given in eq. 34) by setting its derivative with respect to \( m_{\text{new}} \) to 0. After rearranging terms this results in the following expression for calculating model updates \( \Delta m = m_{\text{new}} - m_{\text{old}}: \)
\[
(J^T W_d J + S + A + M) \Delta m = -J^T W_d \Delta d(m_{\text{old}}) - (S + A) m_{\text{old}}.
\] (37)

This is a modified form of the Levenberg–Marquardt scheme, the difference being the added terms for structural and anisotropy constraints (given by matrices \( S \) and \( A \)). The steplength damping term \( M \) is a defining feature for a Levenberg–Marquardt type inversion scheme. The Levenberg–Marquardt approach is generally regarded as suitable for the solution of non-linear and ill-posed problems (Jupp & Vozoff 1975; Jennings & McKeown 1992).

After solving eq. (37) for the model update \( \Delta m \), the model update is added to the old model \( m_{\text{old}} \). Since the inverse problem is non-linear, a number of Levenberg–Marquardt iterations (eq. 37) have to be calculated until a final inversion model is found. The matrix equation for model updates is solved with pre-conditioned conjugate gradients (using the same solver as for the forward problem). Again, this has the advantage that the matrices are never explicitly formed and stored in memory. The elements of the matrix are assembled when they are needed. A full description of the algorithm, covering efficient calculation of the Jacobian and computational issues (including parallelization), is given in Pain et al. (2002, 2003).

Note, that the matrices \( S, A \) and \( M \) contain penalty levels that need to be chosen. The penalty levels for structural constraints \( \lambda_{\mu}^s \) and for anisotropy constraints \( \lambda_{\mu}^a \) dictate to what degree structure and/or anisotropic material properties are created in an inversion model. The influence of the penalty level on the inversion result is discussed in Sections 4.1 and 5.1 and guidelines on the choice of numerical values are given. The matrix \( M \) controls the steplength of the model update. It is customary in the Levenberg–Marquardt algorithm to adjust the penalty levels \( \lambda_{\mu}^a \) automatically according to the convergence properties of the solution at each iteration (Bishop 1995). Initial values for \( \lambda_{\mu}^a \) are chosen and the error according to \( F \) in eq. (34) is monitored after each iteration. If
the error decreases after one iteration, the values for $\lambda^f_m$ are decreased by a factor of 10 (allowing larger steplengths). If, however, the error increases, then the $\lambda^f_m$ are increased by a factor of 10 (decreasing the steplength of the model update), the old model is restored and a new model update is computed. This is repeated until a decrease in $F$ is obtained. The Levenberg–Marquardt iterations are terminated if either a pre-determined level of data misfit (measured by $F_2$, eq. 21) or a maximum number of allowed iterations is reached.

3 FIELD SITE, EXPERIMENTAL SET-UP AND MODEL DISCRETIZATION

The development of the presented anisotropic resistivity inversion algorithm was triggered by the need to invert a cross-well data set gathered at a test site in Cornwall, southwest England. Isotropic inversion images were consistently either unreasonably banded or, when imposing higher levels of smoothness penalties, were not able to predict the data (Herwanger 2001). The unreasonable banding is in fact a reliable indicator of the presence of anisotropy. In this section we summarize the geological setting at the test site and describe the survey geometry and the discretization of the experiment by FE.

3.1 Survey specification

In 1998 electrical and seismic cross-well experiments were carried out at the Reskajeage Quarry hydrological test site with the aim of studying the potential of geophysical tomography to predict hydrological parameters in a hard-rock environment. The dominant lithology at the test site comprises a variable succession of medium to dark grey slates with silty laminae, thinly bedded fine-grained turbiditic sandstones, fine-grained massive sandstones and structureless black mudstones (Jefferies et al. 2000). A simplified lithological illustration of the rocks encountered in the two holes used for the geophysical tomography (BH19 and BH20) is given in Herwanger et al. (2004). Bedding planes dip at an angle of 10–15° to the southeast, which is the approximate direction from BH20 towards BH19 (Fig. 2).

Evidence of the presence of abundant open fractures, in addition to many sealed predominantly quartz cemented veins, has been obtained from fully oriented core and impression packing. The predominant fracture set strikes northeast–southwest with gentle or moderate dip to the southeast. These fracture planes cut the bedding planes at angles of 10–20°. The distribution of open fractures is heterogeneous. Zones of intense fracturing (eight to ten open fractures per metre) are evident between intervals where the rock is very sparsely fractured (zero to two open fractures per metre). A sandstone unit between 50 and 70 m depth in BH20 is particularly sparsely fractured.

At this site layering, fracturing and/or the alignment of elongated particles composing the rock can be the cause of the observed anisotropy. In this paper we solely discuss including anisotropy explicitly into electrical inversion. We do not attempt to explain the mechanism causing the observed anisotropy.

3.2 Experimental set-up

Fig. 2 depicts the experimental set-up for both, the electrical experiment at Reskajeage Quarry and for the synthetic data experiment. The data are acquired in pole–pole geometry with 88 source–receiver locations in BH19 and 87 receiver locations in BH20, both at a 1 m interval, resulting in 7656 observed transfer resistances. The covered depth interval is 21–108 m and the well spacing is 25.7 m. Due to access restrictions and safety concerns, the remote electrodes were located at a distance of 22 m from the well heads and cannot therefore be considered to be placed at infinity and need to be modelled. Additionally, we repeated the experiment with current and potential electrodes reversed. Due to the reciprocity theorem (Jackson 1962), identical transfer resistances are observed when sources and receivers are exchanged. Deviations from reciprocity are one indication of data error and are used to choose inversion weights $w_i$ in eq. (20).

3.3 Model discretization

We use a 3-D FE mesh to discretize the subsurface. In the region of interest between the two wells (area shaded grey in Fig. 2) the element-size is roughly $1.5 \times 1.5 \times 1.5$ m. Outside the region of interest the mesh coarsens exponentially towards the boundary of the model domain. The modelling domain extends 150 m beyond the region of interest in all directions, except towards the top, where the Earth’s surface bounds the modelling domain. The total number of nodes of the employed FE mesh is 42,840. At every node point we describe the conductivity tensor by Euler angles $\alpha$, $\beta$ and $\gamma$ and the logarithms of the eigenvalues of the conductivity tensor $\log(\hat{\sigma}_1)$, $\log(\hat{\sigma}_2)$ and $\log(\hat{\sigma}_3)$. The total number of model parameters is thus $6 \times 42,840 = 257,040$.

4 TOMOGRAPHIC INVERSION OF SYNTHETIC DATA

In this section we present inversion results using computer-generated data from a 2-D subsurface model exhibiting similar features and conductivity contrasts to those expected at the Reskajeage field site. The model was created on the basis of an interpretation of electric and seismic tomograms from the field site (see Fig. 10) and the available geological log. The synthetic model is depicted in Figs 3(a)–(c). At the top, a conductive, highly anisotropic zone ($\hat{\sigma}_1 = \hat{\sigma}_2 = 0.005 \, \text{S m}^{-1} \equiv 200 \, \Omega \, \text{m}, \hat{\sigma}_3 = 0.0008 \, \text{S m}^{-1} \equiv 1250 \, \Omega \, \text{m}, \beta = 15^\circ$) is encountered, underlain by a resistive and isotropic zone ($\hat{\sigma}_1 = \hat{\sigma}_2 = \hat{\sigma}_3 = 0.0005 \, \text{S m}^{-1} \equiv 2000 \, \Omega \, \text{m}$). The interface between these two zones is inclined at 45° in the left part of the model and horizontal in the right part. The bottom zone is moderately anisotropic ($\hat{\sigma}_1 = \hat{\sigma}_2 = 0.002 \, \text{S m}^{-1} \equiv 500 \, \Omega \, \text{m}, \hat{\sigma}_3 = 0.001 \, \text{S m}^{-1} \equiv 1000 \, \Omega \, \text{m}, \beta = 7.5^\circ$) and is divided by a moderately conductive, isotropic layer ($\hat{\sigma}_1 = \hat{\sigma}_2 = \hat{\sigma}_3 = 0.0025 \, \text{S m}^{-1} \equiv 400 \, \Omega \, \text{m}$). The Euler angles $\alpha$ and $\gamma$, describing rotations about the z-axis, are zero in the entire domain. Data from this model were simulated using the same source and receiver positions as in the Reskajeage field experiment.

The displayed portion of the model shows the region of interest between the two boreholes, shaded grey in Fig. 2. This portion of the model is also used to display the inversion images.

4.1 Effect of regularization parameters

Levels of roughness penalty parameters $\lambda^r_1$, $\ldots$, $\lambda^r_4$ and anisotropy penalty parameter $\lambda^a$ need to be chosen at each iteration of the solution of the inverse problem. In general, as the magnitudes of $\lambda^r_1$, $\ldots$, $\lambda^r_3$ and $\lambda^a$ are decreased the data fit improves. However, using values that are too small will result in overfitting of the data and backprojection of data errors into the inversion model. Numerous ways...
to estimate appropriate levels of penalty parameters exist (Hansen 1998), with the L-curve method being probably most widely applied. Most of these methods require knowledge of data error and usually assume uncorrelated Gaussian data error statistics. In practice, data error is hard to quantify and the data errors do not exhibit Gaussian statistics. For example, data error can be caused by erroneous electrode positions or unusually large contact resistance of a specific electrode. Both sources of error result in correlated data error and the resulting error statistics can be distinctly non-Gaussian. Pratt & Chapman (1992) suggest careful examination of a suite of solutions, calculated using a range of penalty levels, and the use of maps of data residuals (amongst other indicators) to estimate
optimal regularization parameters. This is also the approach taken here.

In a first test we run 16 inversions, each with fixed penalty levels $\lambda_1^s = \lambda_2^s = \cdots = \lambda_s^s$ and $\lambda^s$ for structure and anisotropy regularization, respectively. Fig. 4 summarizes the penalty parameter levels used and lists the resulting values of the error functional resulting from each inversion. All tests were run with a fixed number of Levenberg–Marquardt iterations of 10 and using a homogeneous, anisotropic starting model. The parameters characterizing the starting model are $\hat{\sigma}_1 = \hat{\sigma}_2 = 0.0025 \, \text{S} \, \text{m}^{-1} (= 400 \, \Omega \, \text{m})$, $\hat{\sigma}_3 = 0.00067 \, \text{S} \, \text{m}^{-1} (= 1500 \, \Omega \, \text{m})$ and $\beta = -11.5^\circ$. This model is derived from fitting an anisotropic half-space model to the data.

If previous knowledge about the magnitude and distribution of conductivity values is available, the relative amplitudes of the six structure penalty levels can be chosen accordingly. However, in this test all six structure penalties $\lambda_1^s, \ldots, \lambda_s^s$ are given the same value. This is a reasonable choice, since rotation angles can range from $-\pi/2$ to $\pi/2$ and the logarithms of common Earth material conductivities range from 4 to 1 (equivalent to resistivities of 10 000 $\Omega \, \text{m}$ to 10 $\Omega \, \text{m}$). Therefore variations in rotation angles and the logarithms of conductivity eigenvalues can be expected to be of similar magnitude and the corresponding structural penalty levels ought to be equal in magnitude. It turns out that the use of the same penalty levels for angles and eigenvalues causes the images of reconstructed angles to exhibit relatively stronger and higher frequency spatial variations compared with images of conductivity eigenvalues. In further inversion tests (see Section 4.3) this is taken into account by using larger penalty parameters for $\alpha$, $\beta$ and $\gamma$ than for $\log(\hat{\sigma}_1)$, $\log(\hat{\sigma}_2)$ and $\log(\hat{\sigma}_3)$.

Since the survey geometry is essentially 2-D (in the $x$–$z$ plane), sensitivities towards rotation of the conductivity tensor around the $z$-axis, measured by the angles $\alpha$ and $\gamma$, are zero. To further discourage variations in these angles, only small initial steplengths are allowed for $\alpha$ and $\gamma$. This is achieved by choosing $\lambda_{11}^s = \lambda_{22}^s = 10^6$ (acting on $\alpha$ and $\gamma$) and setting $\lambda_{12}^s = \lambda_{13}^s = \lambda_{23}^s = 1$ (acting on $\beta$, $\log(\hat{\sigma}_1)$, $\log(\hat{\sigma}_2)$ and $\log(\hat{\sigma}_3)$ respectively).

Fig. 5 displays 16 tomographic inversion images of average conductivity (left) and anisotropy (right) calculated using the parameters given in Fig. 4. The 16 tomograms clearly demonstrate the influence of structural constraints and anisotropy constraints on the inversion images. For large structural penalty levels $\lambda_1^s, \ldots, \lambda_s^s$ (top row, images 1, 2, 3 and 4) the average conductivity and the anisotropy images are structureless and smooth. As the structure penalty level decreases successively more structure develops (cf. rows 2, 3 and 4). In a similar manner, applying a large anisotropy penalty (left column, images 1, 5 and 9) produces nearly isotropic images. Isotropic solutions with low levels of structure penalty (images 9 and 13) show a stripy appearance. The magnitude of the stripy anomalies is large for regions with strong anisotropy and less pronounced in regions of small anisotropy. The stripes are thus an indicator of the presence of unaccounted anisotropy. These stripy artefacts will, when scaled up, create a macroscopically anisotropic medium.

In image 16, the interfaces between regions of constant material properties from the forward model are indicated by white lines. The match between reconstructed average conductivities and anisotropy in the inversion model and the true model (cf. Figs 3b and c) is very encouraging.

Note the non-linear colour scale used for the conductivity image. For the isotropic images (left column, images 9 and 13), the reconstructed conductivities range over two orders of magnitude. For the remainder of the images, the reconstructed values are within the range of $0.001 \, \text{S} \, \text{m}^{-1} = 1000 \, \Omega \, \text{m}$ to $0.005 \, \text{S} \, \text{m}^{-1} = 200 \, \Omega \, \text{m}$. In order to be able to distinguish features in the latter images as well, the colour bar is stretched for average values of conductivity.

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Figure 4. Summary of inversion parameters and resulting values of the error functional for synthetic data inversion.
Further insight into appropriate levels of regularization is gained
by examining a suite of data residual maps. Each of the 16 maps
is colour-coded according to the difference in observed and predicted
transfer resistance for a single source–receiver pair. The pixels are
arranged so that the source depth in BH19 increases from top to
bottom and the receiver depth in BH20 increases from left to right.
Residual maps associated with inversion images using large reg-
ularization parameters exhibit strongly correlated patterns. For ex-
ample, inversion images calculated with large values of anisotropy
penalty level $\lambda^a$, result in residual maps that exhibit diagonal striping
(Fig. 6, left column). The systematic appearance of the residuals can
be readily explained: source–receiver pairs of equal vertical offset
and squeezed for small and large values of conductivity by a tanh
function centred on the average conductivity.

In this inversion test the true subsurface model is known and thus
the best inversion result can be found by comparison of inversion
models with the true model. Optimum regularization parameters can
then be chosen as the ones that yield inversion images that are
closest to the true model. In choosing a synthetic data example that
resembles the expected subsurface structure at a field site a first
indication into the required magnitude of regularization parameters
is gained.

4.2 Strategy for choosing levels of regularization

Further insight into appropriate levels of regularization is gained
by examining a suite of data residual maps. Each of the 16 maps
in Fig. 6 is a display of all the data residuals ($d^{obs}_i - d^{res}_i$) for the
16 solutions shown in Fig. 5. Each pixel in each of these 16 panels
is coloured according to the difference in observed and predicted
model structure as defined by the error functional $F_i$ are chosen) and a measure of data prediction
error (often the logarithm or the square of the root-mean-square

Figure 5. Sixteen synthetic data inversion images, depicting average conductivity (left) and anisotropy (right). Images 1–16 show a suite of inversion results calculated with varying structural penalty levels (decreasing from top to bottom) and anisotropy penalty levels (decreasing from left to right). Values of penalty levels are given in Fig. 4. The effect of constraints can be clearly seen: for large values of structural constraints (top row, images 1, 2, 3 and 4) both average conductivity and anisotropy images are smooth and for large values of anisotropy constraints (left column, images 1, 5, 9 and 13) the anisotropy images show consistently low values of anisotropy. As the penalty levels are relaxed, successively more structure and anisotropy emerges. See text for details.
residual) for a varying regularization parameter. The optimum regularization level is chosen as the point of maximum curvature of the L-shaped curve. In many applications, the point of maximum curvature is chosen by visual inspection of the L-shaped curve, and will in practice depend on the actual definition of the measures employed for model structure and data prediction error and is thus not completely objective. We should also mention here that theoretical concerns have been raised about the validity and applicability of the L-curve method for choosing penalty parameters. For example, Yagola et al. (2002) demonstrate that choosing the regularization parameter with the L-curve method introduces a bias in the solution.

By reducing the information of the residual maps to a single number (the measure of data prediction error), key information about the correlation of data residuals gets lost. On the other hand, information about correlation of residuals is easily accessible when plotting residual maps. For example, the parallel stripes in the large anisotropy penalty maps (two left columns of images in Fig. 6) are strong indicators of too little anisotropy in the inversion model and the need for smaller anisotropy penalty levels. In the same way, plotting of a suite of inversion models allows for the application of arguments of geological plausibility. This opportunity is forfeited when reducing the inversion model to one number (the measure of model structure).

In this section we have shown how regularization parameters nudge inversion models into the direction of models required by the specific regularization operator. By choosing very large penalty parameters the inversion models can be forced to assume the appearance of the end members of the type of model required by the operator, i.e. smooth and/or isotropic models. As a practical way of choosing penalty levels we find the examination of a suite of inversion models calculated using a range of penalty levels and the study of the residual maps associated with the solutions indispensable.

4.3 Optimal inversion strategy?

In geophysical inversion there is usually a need for user input, for example in the choice of a starting model, the level of penalty parameters or even the choice of inversion method. In the opinion of Treitel & Lines (2001) this reads as follows: ‘Perhaps... a future third Millennium review article can report that a machine has been solving the inverse problem without a human arbiter. For the time being, however, what might be called ‘unsupervised geophysical inversion’ remains but a dream.’ In this section we discuss some ideas of human arbitration and relate the resulting experiences.

4.3.1 Choice of starting model

We test the influence of starting models on the inversion result by running two inversions utilizing two different starting models: a homogeneous isotropic model and a homogeneous anisotropic model, both derived from fitting analytical data solutions to the observed data. The two inversions are run with identical inversion parameters. Using the anisotropic starting model a slightly better
match to the known subsurface model can be achieved. However, the inversion images obtained with the isotropic starting model also reproduce all key features of the true subsurface.

4.3.2 Relaxation of penalty levels

In the 16 inversion tests, discussed in Section 4.1, we imposed a fixed number of structural and anisotropy constraints by applying penalty levels that are kept constant at each iteration. A further possible strategy is to initially use large penalty levels for $\lambda_\mu^s$ and relax the penalty levels after each Levenberg–Marquardt iteration by a constant factor (e.g. by a factor of two). Using this strategy in the choice of structural penalty levels, the inversion model at the initial iterations (arguably) consists of the large-scale, long-wavelength features and at later iterations small-scale, short-wavelength features are created. In practice we have found that the difference between fixed $\lambda_\mu^s$ inversion models and relaxing $\lambda_\mu^s$ inversion models is negligible when using noise free data. However, when using noisy data, inversion models using the relaxing $\lambda_\mu^s$ strategy appear smoother at the same data misfit. There is an added benefit in a relaxation scheme for $\lambda_\mu^s$ in that no \textit{a priori} level of $\lambda_\mu^s$ needs to be specified: initially large values for $\lambda_\mu^s$ can be used, the $\lambda_\mu^s$ are relaxed at every iteration and the inversion can be terminated as soon as an acceptable level of data misfit is reached.

In choosing the anisotropy penalty level $\lambda^a$ we find it advisable to use a constant penalty level for all Levenberg–Marquardt iterations, especially when using an anisotropic starting model. Using initially large levels for the anisotropy penalty $\lambda^a$, with successive relaxation, would counteract the use of an anisotropic starting model by forcing isotropic models in initial iterations.

4.3.3 Balancing structural penalty levels and steplengths

For the inversion tests in Section 4.1 we used fixed levels of structure penalties $\lambda_\mu^s$, $\mu = 1, \ldots, 6$ where all six $\lambda_\mu^s$ were set equal. In like manner the steplength penalty levels ($\lambda_\mu^\ell$, $\lambda_4^\ell$, $\lambda_5^\ell$, and $\lambda_6^\ell$) for the rotation angle $\beta$ and the three conductivity eigenvalues were equal to each other. Visual inspection of images of the angles $\beta$ and the three conductivity eigenvalues $\hat{\sigma}_1$, $\hat{\sigma}_2$, $\hat{\sigma}_3$ revealed that the $\beta$ images were generally more serrated than the eigenvalue images.

This leaves the human arbiter with the task of fine-tuning the relative sizes of the $\lambda_\mu^s$ and $\lambda_\mu^\ell$. Since there are a large number of models that predict the observed data equally well, the arbiter has to apply geological and physical plausibility to arrive at a choice of inversion parameters. Accordingly, we chose to increase the structural penalty acting on $\beta$ ($\lambda_2^\ell$) by a factor of 10, resulting in smoother $\beta$ images and at the same time limiting the steplength in the $\beta$ direction by increasing the initial value of $\lambda_2^\ell$ by a factor of 10. Inversion models calculated using these new parameters differ only in detail but not in large-scale structure and general appearance. Thus the inversion models are stable with respect to small changes in inversion parameters and the resulting models are mainly determined by the necessity to fit the observed data and to a lesser degree by the preferences of the human arbiter.

4.3.4 Synthetic data inversion model

In Fig. 3 the known forward model and a typical inversion model (using the fine-tuned parameters from Sections 4.3.1, 4.3.2 and 4.3.3) are compared. The inversion model results from an inversion run using a homogeneous anisotropic starting model, a fixed anisotropy penalty level of $\lambda^a = 1.0 \times 10^{-8}$, using relaxation (by a factor of 2 after each iteration) of structural penalties with initial values of $\lambda_1^s = \lambda_2^s = \lambda_3^s = 5.0 \times 10^{-1}$, $\lambda_4^s = \lambda_5^s = \lambda_6^s = 5.0 \times 10^{-3}$ and initial steplength penalties of $\lambda_1^\ell = \lambda_2^\ell = 1.0 \times 10^6$, $\lambda_3^\ell = 1.0 \times 10^{-3}$, $\lambda_4^\ell = \lambda_5^\ell = \lambda_6^\ell = 1.0 \times 10^{-2}$.

The inversion model is similar to the true model. Figs 3(a) and (d) display the true model and the inversion model respectively, depicting the conductivity tensor in the $x$-$z$ plane as ellipses. The minor and major axes of the ellipse correspond to the values of the eigenvalues $\hat{\sigma}_1$ and $\hat{\sigma}_3$ and the tilt angle of the ellipses images the Euler angle $\beta$. Figs 3(b) and (e) display average conductivity images (with average conductivity defined in eq. (14) for true and inversion models respectively and Figs 3(c) and (f) display anisotropy images for true and inversion models. The definition of percentage anisotropy is given in eq. (15). The location, amplitude and direction of the symmetry axis of all features are well recovered. Interfaces in the inversion model are gradual rather than sharp (as in the true model) due to the applied smoothness constraints. Conductive features are imaged slightly too wide and resistive features are imaged slightly too narrow (cf. Figs 3b and e).

5 TOMOGRAPHIC INVERSION OF FIELD DATA

5.1 Regularization

In this section we present a suite of field data inversion tests, repeating the tests performed on synthetic data in Section 4.1. The field data and synthetic data inversion tests share the same source–receiver geometry, the same FE mesh and exhibit roughly the same conductivity contrasts and large-scale features. This allows us to judge the quality of inversion images achieved from field data inversion in the light of the experience gained from the synthetic data example (where the true subsurface model is known).

Fig. 8 displays 16 inversion models calculated using regularization penalty levels given in Fig. 7 and using a best-fitting half-space model as the starting model. As expected the field data inversions react in a very similar manner to regularization as the synthetic data inversions. For example, note the pronounced striping in the tomograms with large anisotropy penalty levels and small structural penalty levels (images 9 and 13 in Figs 5 and 8). The striping is strong in the entire image in the field data inversions, whereas the major striping is only present in the top part of the synthetic data tomograms. Since these stripes have been identified as artefacts from neglecting anisotropy, one can infer that the subsurface at the field site is markedly anisotropic over the entire study area. A second observation is the markedly more serrated appearance of image 16 in the field data tomogram (Fig. 8) compared with image 16 in the corresponding synthetic data tomogram (Fig. 5). In the calculation of these images the structural penalty level is very small, and thus it stands to reason that some of the high-frequency structure created in the field data tomogram is created by backprojection of noise into the inversion image and the lack of appropriate smoothness constraints to limit the ‘damage’ from backprojection of noise.

The close relationship of inversion results from field data and synthetic data inversions also applies for the residual maps (compare Figs 6 and 9). Correlated residuals of the same general appearance are observed in both figures. However, there is one notable difference: the field data residual maps show a degree of observational noise. The noise is strongly associated with some of the electrodes, evidenced by consistently larger absolute amplitude readings at the
with varying structural penalty levels (decreasing from top to bottom) and anisotropy penalty levels (decreasing from left to right). Values of penalty levels are small structural penalty levels (bottom row, images 15 and 16) the images have a serrated appearance, an effect of backprojection of noise into the model. See Figure 8.

J. V. Herwanger

Figure 7. Summary of inversion parameters and resulting values of the error functional for field data inversion.

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<th>Average Conductivity</th>
<th>Anisotropy</th>
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<td><img src="image1" alt="Average Conductivity" /></td>
<td><img src="image2" alt="Anisotropy" /></td>
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Figure 8. Sixteen field data tomograms depicting average conductivity (left) and anisotropy (right). Images 1–16 show a suite of inversion results calculated with varying structural penalty levels (decreasing from top to bottom) and anisotropy penalty levels (decreasing from left to right). Values of penalty levels are given in Fig. 7. The field data tomograms respond in a very similar manner to structural and anisotropy penalization as the synthetic data tomograms. For very small structural penalty levels (bottom row, images 15 and 16) the images have a serrated appearance, an effect of backprojection of noise into the model. See text for details.
5.2 Anisotropic tomography results

In this section we discuss the inversion images resulting from anisotropic resistivity tomography. Furthermore we compare the resistivity tomographic images with anisotropic velocity images derived from seismic traveltime tomography.

Figs 10(a) and (b) display average conductivity and electrical anisotropy images, respectively. The arrows in the anisotropy image (Fig. 10b) point in the direction of the symmetry axis and their length is proportional to the anisotropy value. Source and receiver positions are denoted by stars and triangles respectively. In like manner, Figs 10(c) and (d) display average velocity and seismic anisotropy images. The seismic images are calculated using a computer program provided by R. G. Pratt (Chapman & Pratt 1992; Pratt & Chapman 1992) and the seismic tomography experiment and parameter choices for the calculation of the seismic tomograms are discussed in Herwanger (2001) and Herwanger et al. (2004).

Anisotropic seismic material properties (velocity or stiffness) are described by a fourth-order tensor and thus cannot be represented by an ellipse. For this reason, we have chosen to indicate the axis of symmetry of the material property tensor by arrows in both electrical and seismic images.

In the shallow part of the boreholes between 20 m and 50 m the largest conductivities (between 0.002 S m$^{-1}$ and 0.003 S m$^{-1}$) are encountered. The electrical anisotropy image for this region shows a strongly anisotropic band dipping from left to right with anisotropy values of 300 per cent. This sequence corresponds geologically to a fractured unit of laminated siltstones. Seismically this sequence expresses itself as low-velocity, high-anisotropy region with velocities of around 4 km s$^{-1}$ and anisotropy of up to 35 per cent (measured by the Thomsen parameter $\epsilon$) (Thomsen 1986). The siltstone unit is underlain by a sandstone body, extending from 47 m to 70 m in the left borehole, thinning towards the right and intersecting the right borehole between 65 m and 70 m. Electrically the sandstone body expresses itself through low conductivities (high resistivities) and small anisotropy. This resistive sandstone body is imaged too thin in the conductivity image (when compared with the seismic image). A similar observation has already been made in the inversion of the synthetic data, where the resistive body is ‘squeezed’ in the inversion image (cf. Figs 3b and e). Seismically the sandstone unit is characterized by high velocity and low anisotropy. Below 70 m the tomographic images become more complex; however, the structures imaged by electric and seismic tomography still broadly correlate: zones of high conductivity consistently show low velocities and vice versa, zones of low conductivity express themselves by high velocities. A remarkable correlation is observed for the estimates of angle of the symmetry axis in the electric and seismic images: in the upper part of the anisotropy images (see Figs 10b and d) the angles vary at around 15° and in the lower part of the model angles of around 5° predominate.
6 DISCUSSION

We have presented synthetic and field data inversions, reconstructing the electrical conductivity tensor at each node point of a FE mesh. Using synthetic data (from a known subsurface model) we have shown that it is feasible to reconstruct meaningful anisotropic conductivity images from electrical cross-well data. The key to the solution of the ill-conditioned system of tomography equations is the use of appropriate regularization. We use a regularization scheme that includes penalty functionals for model roughness and anisotropy. Additionally, step-length damping is used to improve the conditioning of the number of the system of tomography equations. Using synthetic and field data we have shown the influence and the importance of the penalty functionals on the character of the solution and the ability of the reconstruction algorithm to reconstruct key features of the true structure. In order to choose appropriate penalty levels we have found it essential to examine a suite of solutions with varying structural and anisotropy constraints. Furthermore, maps of the data residuals provide invaluable assistance in choosing appropriate penalty levels.

In the inversion of field data we have found (average) conductivity values between 0.005 S m$^{-2}$ ($=200$ $\Omega$ m) and 0.001 S m$^{-1}$ ($=1000$ $\Omega$ m) with laterally strongly varying anisotropy between 0 and 300 per cent. The strongest anisotropy is encountered in a dipping band of finely laminated and fractured siltstone. The underlying sandstone body was found to be highly resistive and isotropic.

Failure to include anisotropy in the inversion of the field data results in subsurface images that are severely distorted by artefacts. The strength of the artefacts renders the inversion models uninterpretable and the inversion model does not contain useful structural information on the subsurface. Reconstruction of a heterogeneous anisotropic subsurface was previously not possible. Thus anisotropic resistivity inversion is a major extension to research in electrical imaging and is an important contribution to electrical imaging applications.

The ability of the presented algorithm to recover features of the true subsurface model is further confirmed by the similarity of anisotropic seismic and electrical inversion images. Highly resistive regions exhibit high seismic velocities and electrically highly anisotropic areas are also seismically anisotropic. Under the assumption of a transversely isotropic medium, the directions of the symmetry axes of electrical and seismic anisotropy correspond well, with an inclination of the symmetry axis of about 15° in the shallow part of the model and about 5° in the bottom part of the model.

At present we use smoothness constraints for structural regularization. Geological materials are seldom smooth and therefore a priori information taking the subsurface statistical properties into account could be more suited to regularization purposes. These priors could, for example, be derived from well logs or seismic images. The inclusion of different forms of structural priors (Haber & Oldenburg 1997; Kaipio et al. 1999) is expected to be the next big improvement in the development of electrical tomography. The use of structure of the seismic tomograms to constrain the electrical tomogram would be a straightforward application of sequential inversion (e.g. Lines et al. 1988).

Harnessing the ability of the presented inversion method to reconstruct the spatial variation of tensor conductivity, the next obvious step is to relate anisotropic electrical conductivity to (anisotropic) fluid flow properties. It has been shown, for both groundwater and oil-field applications, that the resistivity tensor can have a close relationship to anisotropic pore space (including fracture) properties (e.g. Ritzi & Andolsek 1992; Schön et al. 1999), and our present research is directed at integrating (anisotropic) seismic and electrical tomographic methods for the estimation of fluid flow characteristics in the subsurface. We believe that anisotropic electrical images can play an important role in reservoir engineering and groundwater monitoring when using fixed sensors in oil and water wells.

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